Consider a rigid body comprising of n particles having masses m_a ($a = 1, 2, \ldots, n$).

Let one of the points be fixed.

$\therefore$ translational motion is absent

System rotates with instantaneous angular velocity $\vec{\omega}$.

$$\Rightarrow \quad \vec{v}_a = \vec{\omega} \times \vec{r}_a .$$

$$\therefore \vec{L} = \sum_{a=1}^{n} \vec{l}_a = \sum_a \vec{r}_a \times m_a \vec{v}_a .$$

$$= \sum_a m_a \vec{r}_a \times (\vec{\omega} \times \vec{r}_a).$$

$$= \sum_a m_a r_a^2 \vec{\omega} - \sum_a m_a (\vec{r}_a \cdot \vec{\omega}) \vec{r}_a .$$

e.g.,
$$L_x = \sum_a m_a (r_a^2 - x_a^2) \omega_x - \sum_a m_a x_a y_a \omega_y - \sum_a m_a x_a z_a \omega_z .$$

Moments & products of inertia.

$$\vec{L} = \overleftrightarrow{I}\,\vec{\omega}$$

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

$$I_{xx} = \sum_a m_a (r_a^2 - x_a^2) = \sum m_a (y_a^2 + z_a^2)$$

$$= \int \rho(\vec{r})(r^2 - x^2)\,d\tau$$

continuum version.

$$I_{yy} = \sum_a m_a (z_a^2 + x_a^2)$$

$$I_{zz} = \sum_a m_a (x_a^2 + y_a^2)$$

$$I_{xy} = -\sum_a m_a x_a y_a = I_{yx}, \quad = -\int \rho\, xy\, d\tau$$

$$I_{yz} = -\sum_a m_a y_a z_a = I_{zy}.$$

$$I_{zx} = -\sum_a m_a z_a x_a = I_{xz}.$$

In compact notation,

$$L_i = \sum_j I_{ij}\,\omega_j \quad , j = 1, 2, 3.$$

$$2T = 2\sum \tfrac{1}{2} m_a |\vec{v}_a|^2 = \sum_a m_a (\vec{\omega}\times\vec{r}_a)\cdot(\vec{\omega}\times\vec{r}_a)$$

$$= \sum_a m_a \vec{\omega}\cdot[\vec{r}_a \times (\vec{\omega}\times\vec{r}_a)].$$

$$= \vec{\omega}\cdot\sum_a m_a \vec{r}_a \times (\vec{\omega}\times\vec{r}_a)$$

$$= \vec{\omega}\cdot\vec{L} \qquad 2T = \vec{\omega}\cdot\vec{L}.$$

$$T = \frac{1}{2}\vec{\omega}\cdot\vec{L}$$

$$= \frac{1}{2}\sum_i \omega_i L_i$$

$$= \frac{1}{2}\sum_{ij} I_{ij}\,\omega_i\,\omega_j \,.$$

$$\vec{\tau} = \left(\frac{d\vec{L}}{dt}\right)_{fix}$$

$$= \left(\frac{d\vec{L}}{dt}\right)_{rot} + \vec{\omega}\times\vec{L}$$

$$\left(\frac{d\vec{L}}{dt}\right)_{rot} = \left(\frac{d\vec{L}}{dt}\right)_{body} = \left[\frac{d}{dt}\left(\overleftrightarrow{I}\cdot\vec{\omega}\right)\right]_{body}$$

$$= \overleftrightarrow{I}\cdot\dot{\vec{\omega}} \,.$$

$$\therefore\ \vec{\tau} = \overleftrightarrow{I}\cdot\dot{\vec{\omega}} + \vec{\omega}\times\vec{L} \,.$$

If we orient the axes of the body frame of reference such that they coincide with the principal axes of the body, all the products of inertia vanish.

$$\left.\begin{aligned} \tau_1 &= I_1\dot{\omega}_1 + (I_3 - I_2)\,\omega_2\omega_3 \\ \tau_2 &= I_2\dot{\omega}_2 + (I_1 - I_3)\,\omega_3\omega_1 \\ \tau_3 &= I_3\dot{\omega}_3 + (I_2 - I_1)\,\omega_1\omega_2 \end{aligned}\right\}\ \text{Euler's eq.}$$

$$\boxed{\text{Show, } \vec{\omega}\cdot\bar{\tau} = \frac{dT}{dt}}$$

Torque free motion.

Free symmetric top

$$I_1 = I_2 \neq I_3.$$

$$\tau_i = 0 \quad \forall i.$$

$$\therefore \quad I_1\dot{\omega}_1 = (I_1 - I_3)\,\omega_2\omega_3.$$

$$I_2\dot{\omega}_2 = (I_1 - I_3)\,\omega_3\omega_1.$$

$$I_3\dot{\omega}_3 = 0.$$

$$\therefore \quad \omega_3 = \text{const}.$$

$$\dot{\omega}_1 = \left[\frac{(I_1 - I_3)\,\omega_3}{I_1}\right]\omega_2.$$

i.e., $\dot{\omega}_1 = \Omega\omega_2.$

& $\dot{\omega}_2 = -\Omega\omega_1.$

where, $\Omega = \left(\frac{I_1 - I_3}{I_1}\right)\omega_3 = \text{const}.$

$\ddot{\omega}_1 = -\Omega^2\omega_1. \quad \omega_1 = A\sin(\Omega t + \theta_0).$

L12

5

$$\omega_2 = A\cos(\Omega t + \theta_0). \quad \text{(Integrating)}$$

$$\therefore \quad \vec{\omega}_p = \hat{i}\,\omega_1 + \hat{j}\,\omega_2 .$$