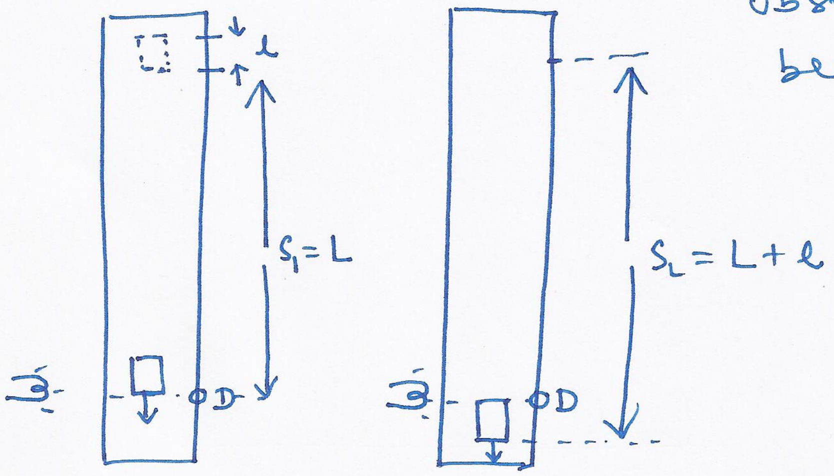
Preliminaries

Mathematical approx. methods

Motivation : To keep the physical behavior tractable when mathematical expressions are complicated.

(Note: more discussion in class)

Setup to determine g: Measure time the object obstructs the light beam.



For a freely falling body starting from rest, the distance s travelled in time t is

$$s = \frac{1}{2} g t^2 .$$

$$\Rightarrow \quad t = \sqrt{\frac{2}{g}} \sqrt{s} .$$

The time interval for the body to fall from $S_1 = L$ to $S_2 = L + l$ is

$$t_2 - t_1 = \sqrt{\frac{2}{g}} \left(\sqrt{S_2} - \sqrt{S_1} \right) .$$

$$= \sqrt{\frac{2}{g}} \left(\sqrt{L+l} - \sqrt{L} \right) .$$

(Ref: Kleppner & Kolenkow)

If $t_2 - t_1$ is measured experimentally, g is given by

$$g = 2\left(\frac{\sqrt{L+l} - \sqrt{L}}{(t_2 - t_1)}\right)^2.$$

This formula is exact but may not be the most useful expression for our purpose.

Consider the factor

$$\sqrt{L+l} - \sqrt{L}$$

in practice $L \gg l$

e.g., $L = 1\,m$, $l = 0.01\,m$.

(above expn. difficult to obtain without using a calculator!).

$\therefore$ Use "power series expansion"

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \dots$$

for $-1 < x < 1$.

Obtained formally soon.

Error incurred on terminating this approximation at some point is of the order of first neglected term.

$$\therefore \sqrt{L+l} - \sqrt{L} = \sqrt{L}\left(\sqrt{1+\frac{l}{L}} - 1\right)$$

$$= \sqrt{L}\left[1 + \frac{1}{2}\left(\frac{l}{L}\right) - \frac{1}{8}\left(\frac{l}{L}\right)^2 + \frac{1}{16}\left(\frac{l}{L}\right)^3 + \dots - 1\right].$$

$$\cancel{= \sqrt{L}\left[1 + \frac{1}{2}\left(\frac{l}{}\right.\right.}$$

$$= \sqrt{L}\left[\frac{1}{2}\left(\frac{l}{L}\right) - \frac{1}{8}\left(\frac{l}{L}\right)^2 + \frac{1}{16}\left(\frac{l}{L}\right)^3 + \dots\right].$$

$$= \sqrt{L}\,\frac{1}{2}\left(\frac{l}{L}\right)\left[1 - \frac{1}{4}\left(\frac{l}{L}\right) + \frac{1}{8}\left(\frac{l}{L}\right)^2 + \dots\right].$$

$$= \frac{l}{2\sqrt{L}}\left[1 - \frac{1}{4}\left(\frac{l}{L}\right) + \frac{1}{8}\left(\frac{l}{L}\right)^2 + \dots\right].$$

"Useful form"

Note: No gain if t can be measured accurate to only 1 part in 1000 by keeping terms beyond second term.

# of terms based on acc. required

The first factor $\frac{l}{2\sqrt{L}}$ gives the dominant behavior.

If $\frac{l}{L} = 0.01$

then, $\frac{1}{8}\left(\frac{l}{L}\right)^2 = 1.2 \times 10^{-5}$.

$\therefore$ If we terminate the above series at second term, error is 1 part in 10^5.

Binomial series

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots + \frac{n(n-1)\dots(n-k+1)}{k!}x^k + \dots$$

for $-1 < x < 1$, $\forall n$.

For $n = \frac{1}{2}$

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \dots \qquad -1<x<1$$

$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \mathcal{O}(x^3).$$

order of x^3.

If $x > 1$, the series can be applied as follows, $(1+x)^n = x^n\left(1+\frac{1}{x}\right)^n$.

$$= x^n\left[1 + n\frac{1}{x} + \frac{n(n-1)}{2!}\left(\frac{1}{x}\right)^2 + \dots\right].$$

Examples: (1.) $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots$

$-1<x<1$.

(2.) $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$

$-1<x<1$.

(3.) $(1001)^{1/3} = (1000)^{1/3}(1+0.001)^{1/3}$

$= (1000)^{1/3}\left(1 + 0.001\left(\frac{1}{3}\right) + \dots\right) \approx 10(1.0003)$

$= 10.003$.

Taylor series

Arbitrary function $f(x)$ can be represented by power series in x;

$$f(x) = a_0 + a_1 x + a_2 x^2 + \dots = \sum_{k=0}^{\infty} a_k x^k .$$

$$f(0) = a_0 .$$

If the function is differentiable to any order,

$$\frac{df}{dx} = f^{(1)}(x) = a_1 + 2a_2 x + \dots$$

$$\therefore \; a_1 = f^{(1)}(x)\Big|_{x=0} .$$

Similarly,

$$a_k = \frac{1}{k!} f^{(k)}(x)\Big|_{x=0} .$$

$f^{(k)}(x)$ is the k^{th} derivative of $f(x)$.

$\therefore$ Taylor series is obtained as,

$$f(x) = f(0) + f^{(1)}(0)\, x + f^{(2)}(0)\, \frac{x^2}{2!} + \dots$$

In general,

$$f(x+a) = f(a) + f^{(1)}(a)\, x + f^{(2)}(a)\, \frac{x^2}{2!} + \dots$$

Behavior of function in neighborhood of point a, Let $f(x) = \sin x$.

$f(0) = \sin(0) = 0$.

$f^{(1)}(0) = \cos(0) = 1$.

$f^{(2)}(0) = -\sin(0) = 0$.

$f^{(3)}(0) = -\cos(0) = -1$.

$\vdots$

$\therefore \sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots$

Similarly

$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \dots$

[Check: $\cos x + i\sin x = e^{ix}$.].