Preliminaries

Vectors (Recapitulation)

Notation:

$\vec{A}$: vector A

$\hat{A}$: unit vector along $\vec{A}$.

$|\vec{A}|$: magnitude of vector $\vec{A}$.

Q. Why do we need quantities like vectors (or for that matter Tensors).

Ans. To handle physical quantities conveniently.

Consider an equation

$$\underbrace{J}_{\text{current density}} = \underbrace{\sigma}_{\text{conductivity}} \underbrace{E}_{\text{Electric field}}.$$

vs

$$\underbrace{\vec{J}}_{\text{current density vector}} = \underbrace{\overset{\leftrightarrow}{\sigma}}_{\text{conductivity tensor}} \underbrace{\vec{E}}_{\text{Electric field vector}}.$$

* example: $\vec{E}$ in x-y plane but $\vec{J}$ along z-axis.

$$\begin{pmatrix} 0 \\ 0 \\ J_z \end{pmatrix} = \begin{pmatrix} 0 & 0 & \sigma_1 \\ 0 & 0 & \sigma_2 \\ \sigma_3 & \sigma_4 & \sigma_5 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ 0 \end{pmatrix}$$

(+ other examples mentioned in class.)

Orthonormal basis for a vector space.

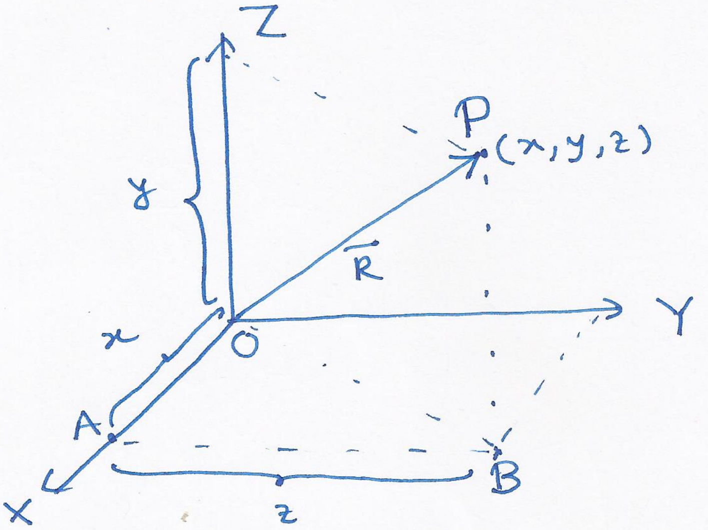
Example : $\hat{i}, \hat{j}, \hat{k}$ forms an orthonormal basis. Here $\hat{i}, \hat{j}, \hat{k}$ are mutually perpendicular unit vectors.

$$\left\{ \begin{array}{l} \hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1 \ , \ \hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0 \\ \hat{i}\times\hat{j} = \hat{k} \ , \ \hat{j}\times\hat{k} = i \ , \ \hat{k}\times\hat{i} = \hat{j} \ ; \ \hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = 0 . \end{array} \right.$$

Position vector : The position of a physical object is completely specified by its position vector. It is a vector drawn from the origin of the coordinate system to the object in question.

Resolution of a vector in cartesian coordinates

$$\vec{OP} = \vec{R} = \vec{OA} + \vec{AB} + \vec{BP} = x\hat{i} + y\hat{j} + z\hat{k} .$$



$$|\vec{OP}| = |\vec{R}| = \sqrt{(\vec{OA})^2 + (\vec{AB})^2 + (\vec{BP})^2} .$$

$$= \sqrt{x^2 + y^2 + z^2} .$$

Rules / Definitions

* Addition of vectors

$$\vec{A} + \vec{B} = (A_x \hat{i} + A_y \hat{j} + A_z \hat{k}) + (B_x \hat{i} + B_y \hat{j} + B_z \hat{k})$$

$$= (A_x + B_x)\hat{i} + (A_y + B_y)\hat{j} + (A_z + B_z)\hat{k}.$$

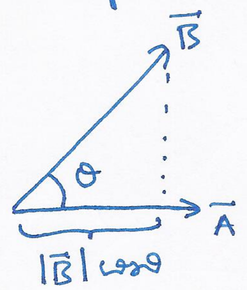
* Scalar product (dot product)

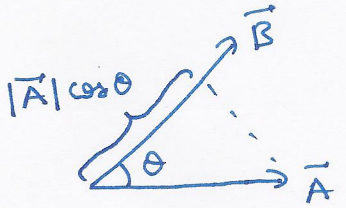
$$\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos\theta .$$

$$= A_x B_x + A_y B_y + A_z B_z .$$

θ is the angle between $\vec{A}$ & $\vec{B}$ when they are placed tail to tail).

Physically, the scalar product $\vec{A} \cdot \vec{B}$ equals the product of $|\vec{A}|$ and the component of $\vec{B}$ along $\vec{A}$. (or, the product of $|\vec{B}|$ and the component of $\vec{A}$ along $\vec{B}$).





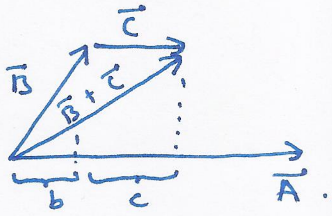
Note: • scalar product is commutative.

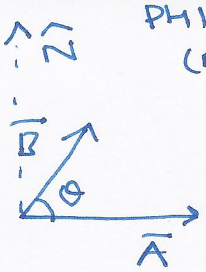
Example:
Work
$W = \vec{F} \cdot \vec{D}$
Force
Displacement

$$\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A} .$$

• $|\vec{A}|(b + c) = b|\vec{A}| + c|\vec{A}|$.

• $\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$.





* <u>vector product</u>

$$\vec{A} \times \vec{B} = |\vec{A}||\vec{B}| \sin\theta \, \hat{N}$$

$$= (A_x\hat{i} + A_y\hat{j} + A_z\hat{k}) \times (B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$$

$$= (A_y B_z - A_z B_y)\hat{i} + (A_z B_x - A_x B_z)\hat{j} + (A_x B_y - A_y B_x)\hat{k}.$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Unit vector $\hat{N}$ is $\perp$ to the plane containing $\vec{A}$ & $\vec{B}$ such that a right handed screw when rotated from direction of $\vec{A}$ to direction of $\vec{B}$, moves forward.

Torque $\vec{\tau} = \vec{R} \times \vec{F}$

(We will understand more about the vectors & the their applications in physics (& Engineering) through some example problems which follow).

C1) Given $\vec{A} = (2\hat{i} - 3\hat{j} + 7\hat{k})$

$\vec{B} = (5\hat{i} + \hat{j} + 2\hat{k})$

Find: (i) $\vec{A} + \vec{B}$.

(ii) $\vec{A} - \vec{B}$.

(iii) $\vec{A} \cdot \vec{B}$.

(iv) $\vec{A} \times \vec{B}$.

(v) angle between $\vec{A}$ & $\vec{B}$.

Soln. (i) $\vec{A} + \vec{B} = (2+5)\hat{i} + (-3+1)\hat{j} + (7+2)\hat{k}$

$= 7\hat{i} - 2\hat{j} + 9\hat{k}$.

(ii) $\vec{A} - \vec{B} = (2-5)\hat{i} + (-3-1)\hat{j} + (7-2)\hat{k}$.

$= -3\hat{i} - 4\hat{j} + 5\hat{k}$.

(iii) $\vec{A} \cdot \vec{B} = (2)(5) + (-3)(1) + (7)(2)$

$= 10 - 3 + 14$

$= 21$.

(iv) $\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -3 & 7 \\ 5 & 1 & 2 \end{vmatrix}$

$= -15\hat{i} + 31\hat{j} + 17\hat{k}$.

(v) $|\vec{A}| = \sqrt{2^2 + (-3)^2 + 7^2} = \sqrt{62}$

$|\vec{B}| = \sqrt{5^2 + (1)^2 + 2^2} = \sqrt{30}$

$\therefore \theta = \cos^{-1}\left(\frac{\vec{A} \cdot \vec{B}}{|\vec{A}||\vec{B}|}\right) = \cos^{-1}\left(\frac{21}{\sqrt{62 \times 30}}\right) \simeq 60.86°$.

(C2) Show that if $|\vec{A} - \vec{B}| = |\vec{A} + \vec{B}|$, then $\vec{A}$ is $\perp$ to $\vec{B}$.

Soln. $|\vec{A} - \vec{B}| = |\vec{A} + \vec{B}|$ (given).

$$\therefore (\vec{A} - \vec{B}) \cdot (\vec{A} - \vec{B}) = (\vec{A} + \vec{B}) \cdot (\vec{A} + \vec{B}).$$

$$\Rightarrow |\vec{A}|^2 - 2\vec{A}\cdot\vec{B} + |\vec{B}|^2 = |\vec{A}|^2 + 2\vec{A}\cdot\vec{B} \pm |\vec{B}|^2.$$

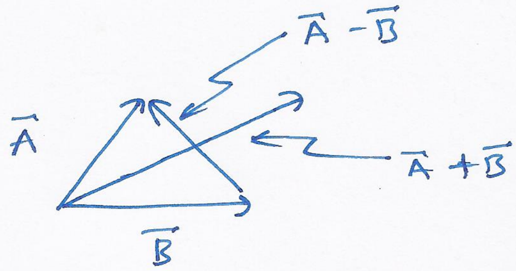
$$\Rightarrow -2\vec{A}\cdot\vec{B} = +2\vec{A}\cdot\vec{B}.$$

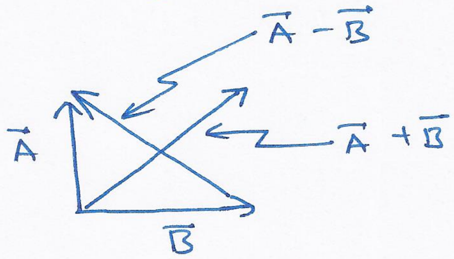
$$\therefore \vec{A}\cdot\vec{B} = 0.$$

$$\Rightarrow \theta = \cos^{-1}(0) = \frac{\pi}{2}.$$

$\therefore \vec{A}$ is $\perp$ to $\vec{B}$.

(Q.E.D.)



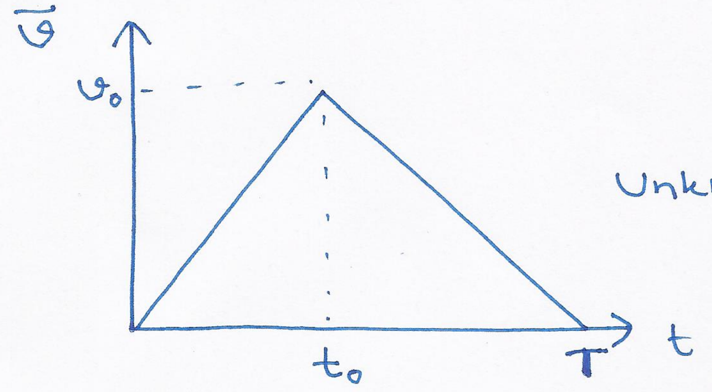


(C3) A sports car can accelerate uniformly to 120 miles/hr in 30 s. Its maximum braking rate cannot exceed 0.7g. What is the minimum time required to go 0.5 mile, assuming it begins & ends at rest?

Known: $a_{accl.}$, $a_{brak.}$, S, t_0 (time for reaching max. speed)

Unknown: T_{min} (to accelerate & come back to rest in a distance 0.5 miles)

Sol^n.



$$v_0 = a_{acc}\, t_0 = a_{brak}\,(T - t_0)$$

$$\therefore\ t_0 = \frac{a_{brak}\, T}{a_{acc} + a_{brak}}$$

But $S = \frac{1}{2} v_0 T$. (area under the curve).

$$\therefore\ S = \frac{1}{2}\, \frac{a_{acc}\, a_{brak}}{(a_{acc} + a_{brak})}\, T^2 .$$

$$\Rightarrow T = \left[\frac{2S\,(a_{acc} + a_{brak})}{a_{acc}\, a_{brak}} \right]^{1/2}$$

$$\left.\begin{aligned} a_{acc.} &= 1.11 \times 10^{-3} \text{ miles/s}^2. \\ a_{brak.} &\leq 4.24 \times 10^{-3} \text{ miles/s}^2. \end{aligned}\right\}$$

T is minimized by making a_{brak} as large as possible.

$\Rightarrow T_{min} = 33.7\ s.$