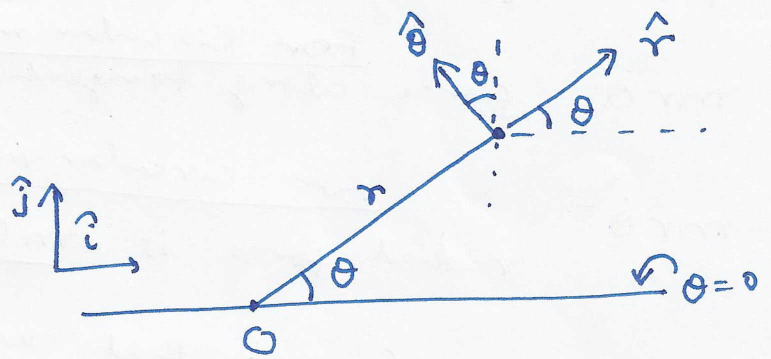
Particle motion in polar coordinates.

PH101
L4 (ADT)



$$\hat{r} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}.$$

Note $\dot{\{\}} = \frac{d}{dt}\{\}$.

$$\hat{\theta} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}.$$

$$\Rightarrow \frac{d\hat{r}}{d\theta} = \hat{\theta}, \quad \frac{d\hat{\theta}}{d\theta} = -\hat{r}.$$

$$\therefore \dot{\hat{r}} = \dot{\theta}\,\hat{\theta}; \quad \dot{\hat{\theta}} = -\dot{\theta}\,\hat{r}.$$

$$\vec{r} = r\,\hat{r}.$$

$$\therefore \vec{v} = \frac{d\vec{r}}{dt} = \dot{\vec{r}} = \dot{r}\,\hat{r} + r\,\dot{\hat{r}}$$

$$\therefore \boxed{\vec{v} = \dot{r}\,\hat{r} + r\dot{\theta}\,\hat{\theta}.}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(\dot{r}\,\hat{r}) + \frac{d}{dt}(r\dot{\theta}\,\hat{\theta}).$$

$$= \ddot{r}\,\hat{r} + \dot{r}\,\dot{\hat{r}} + \dot{r}\dot{\theta}\,\hat{\theta} + r\ddot{\theta}\,\hat{\theta} + r\dot{\theta}\,\dot{\hat{\theta}}.$$

$$= \ddot{r}\,\hat{r} + \dot{r}\dot{\theta}\,\hat{\theta} + \dot{r}\dot{\theta}\,\hat{\theta} + r\ddot{\theta}\,\hat{\theta} - r\dot{\theta}^2\,\hat{r}.$$

$$\therefore \boxed{\vec{a} = (\ddot{r} - r\dot{\theta}^2)\,\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{\theta}.}$$

As $m\ddot{\vec{r}} = \vec{F} \Rightarrow \vec{F} = F_r\,\hat{r} + F_\theta\,\hat{\theta}$

$$F_r = m(\ddot{r} - r\dot{\theta}^2)$$
$$F_\theta = m(2\dot{r}\dot{\theta} + r\ddot{\theta}).$$

For radial motion:

(I.) $m\ddot{r}$ force along radial direction.

For circular motion:

(II.) $mr\ddot{\theta}$ force along tangential direction

For circular motion

(III.) $-mr\dot{\theta}^2$ radial force is $-\frac{m(r\dot{\theta})^2}{r} = -\frac{mv^2}{r}$.

(a force that causes centripetal acceleration).

(IV.) $2m\dot{r}\dot{\theta}$ is not so obvious.

It is associated with the Coriolis force.

(It exists to conserve angular momentum).

Direct consequence

viz, $F_\theta = (mr\ddot{\theta} + 2m\dot{r}\dot{\theta}) = 0$ (say)

Then, $\frac{d}{dt}\underbrace{(mr^2\dot{\theta})}_{L} = 0$.

$\downarrow$

L is conserved

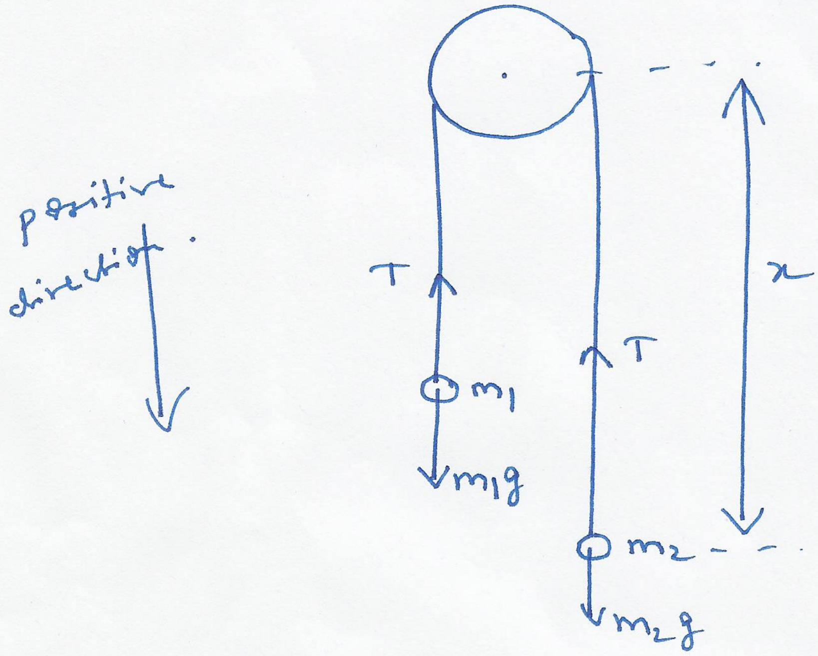
when force is purely radial.

Other examples would be demonstrated

in Tutorial session.

Atwood's machine (I.)

- Light inextensible string of length l.
- Massless pulleys.



- $m_2 > m_1$.

Let position of m_2 be x.

This also fixes position of m_1.

Unknowns: a & T.

As $m_2 > m_1$, $\Rightarrow$ m_2 moves down & m_1 upward.

$$\Rightarrow m_1 \ddot{\vec{x}} = \vec{T} - m_1 \vec{g}.$$

removing vector signs,

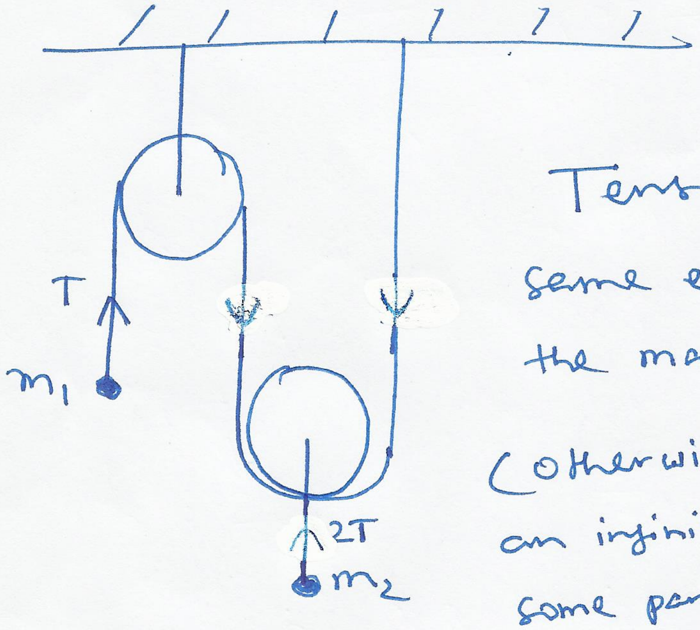
$$\left.\begin{aligned} m_1 \ddot{x} &= T - m_1 g. \\ \text{Also} \quad m_2 \ddot{x} &= -T + m_2 g. \end{aligned}\right\} \text{2 eqn.}$$

$$\therefore\ a = \ddot{x} = \left(\frac{m_2 - m_1}{m_1 + m_2}\right) g.$$

$$\& \quad T = \frac{2 m_1 m_2 g}{m_1 + m_2}.$$

Atwoods machine (II).

→ Massless pulleys.
→ Inextensible strings (massless).



Tension T is the same everywhere throughout the massless string.

(Otherwise there would be an infinite acceleration of some part of the string).

∴ Total tension in the short string connected to m_2 is $2T$.

(∵ zero net force on massless right pulley).

With upward direction taken as positive,

$$\left.\begin{aligned} T - m_1 g &= m_1 a_1 \\ 2T - m_2 g &= m_2 a_2 \end{aligned}\right\} \text{— (A), (B).}$$

2 eqn. & 3 unknowns (T, a_1, a_2).

One more eqn. needed!

Imagine right pulley & mass m_2 move up a distance 'd'.

∴ length 2d of string has to go to the string length to which m_1 is hung. ("conservation" of string).

$$\Rightarrow \quad y_1 = -2y_2 .$$

where, y_1 & y_2 are measured relative to the initial locations of masses m_1 & m_2.

Taking two time derivatives,

$$\Rightarrow \quad a_1 = -2a_2 . \quad \text{—(C)}$$

Using (A), (B) & (C),

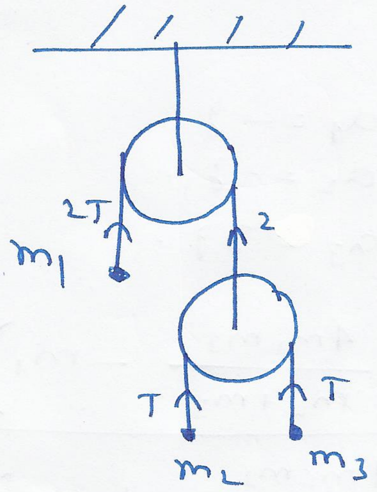
$$\Rightarrow \quad a_1 = g\,\frac{(2m_2 - 4m_1)}{(4m_1 + m_2)}$$

$$a_2 = g\,\frac{(2m_1 - m_2)}{(4m_1 + m_2)}$$

$$\& \quad T = \frac{3m_1 m_2 g}{4m_1 + m_2} .$$

Atwood machine (III.)

Unknowns : a_1, a_2, a_3, T.



Let tension in lower string be T.

$\therefore$ tension in upper string is $2T$.

$$\Rightarrow \left.\begin{aligned} 2T - m_1 g &= m_1 a_1 . \\ T - m_2 g &= m_2 a_2 \\ T - m_3 g &= m_3 a_3 . \end{aligned}\right\} \quad (A), (B), (C)$$

4th eq (?)

Conservation of "string"

$$\Rightarrow a_1 = -\left(\frac{a_2 + a_3}{2}\right) \quad \text{—} (D).$$

(Avg. position of m_2 & m_3 moves the same distance as the bottom pulley, which in turn moves the same distance (but in opposite direction) as m_1).

$$a_1 = g\,\frac{4m_2m_3 - m_1(m_2+m_3)}{4m_2m_3 + m_1(m_2+m_3)}; \qquad a_3 = \frac{-g\left[4m_2m_3 + m_1(m_3 - 3m_2)\right]}{4m_2m_3 + m_1(m_2+m_3)}.$$

$$a_2 = -g\,\frac{4m_2m_3 + m_1(m_2 - 3m_3)}{4m_2m_3 + m_1(m_2+m_3)};$$

(Special)

Limiting cases

(i.) $m_2 = m_3 = \frac{m_1}{2}$. $\Rightarrow$ All a's are zero.

(ii.) $m_3 \ll m_1, m_2$. $\Rightarrow$ $a_1 = -g$.
$a_2 = -g$.
$a_3 = 3g$.

Note $a_1 \overset{\text{form}}{\equiv} \dfrac{g\left(\dfrac{4m_2m_3}{m_2+m_3} - m_1\right)}{\left(\dfrac{4m_2m_3}{m_2+m_3} + m_1\right)}$

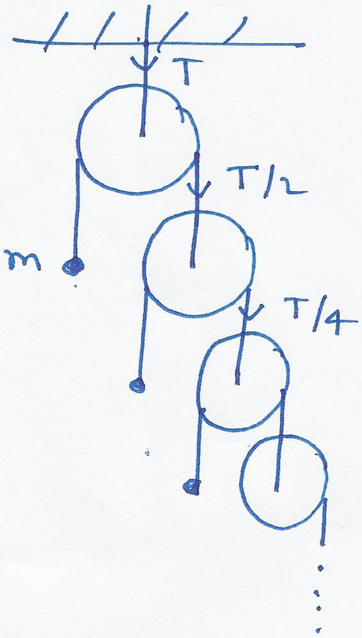
m_2, m_3 pulley system acts like

a mass $\dfrac{4m_2m_3}{(m_2+m_3)}$.

0 (if m_2 or m_3 is zero)

$2m$ (if $m_2 = m_3 = m$).

Infinite Atwood's machine



Accl^n. of top mass = ?

Let accl^n. of top mass be a.

$$\frac{T}{g} = \frac{T/2}{(g-a)}$$

(Argument explained in class.
second pulley lives in a
world with accl^n. due to
gravity
$(g - a_2)$.) .

$$\Rightarrow \quad a_2 = \frac{g}{2}$$