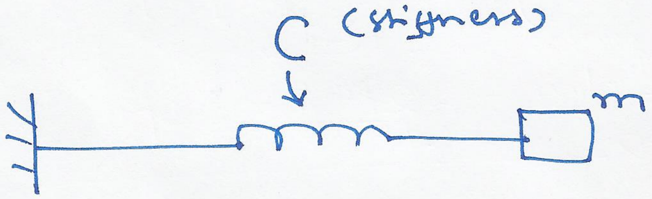
Classical realizations of Harmonic oscillator.

- Mass on a spring in the limit of small amplitude
- LC circuit for "small enough currents"! (s.t., ckt. elem. are linear)
- Simple pendulum for small angle of osillations.

Important properties of Harmonic oscillator

- f is independent of amplitude (in linear regime).
- Effects of several driving forces can be superimposed linearly.



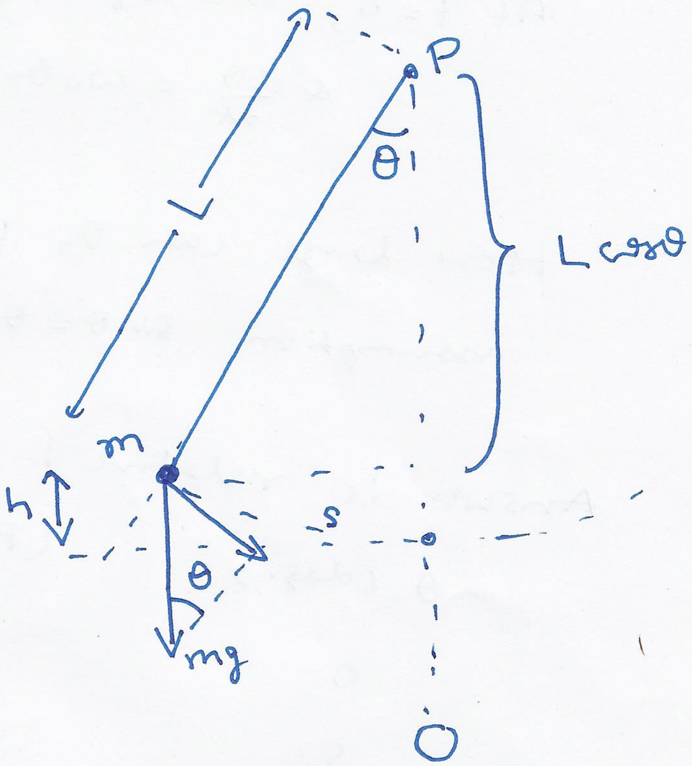
$$\bar{F} = -Cx\hat{x}.$$

x is positive for stretch & negative for compression.

i.e., $M\dfrac{d^2x}{dt^2} = -Cx.$

$\therefore \quad x = A\sin(\omega_0 t + \phi).$

$\omega_0 = \left(\frac{C}{M}\right)^{1/2}$; At $t=0$, $x = x_0 = A\sin\phi$; $\frac{dx}{dt} = v_0 = \omega_0 A\cos\phi.$



$$S = L\theta$$

$$\therefore\ v = \frac{ds}{dt} = L\frac{d\theta}{dt} = L\dot{\theta}.$$

$$a = \frac{d^2s}{dt^2} = L\ddot{\theta}.$$

$\vec{F} = m\vec{a}$ reduces to

$$mg\sin\theta = -mL\frac{d^2\theta}{dt^2}.$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \dots$$

For small θ, $\sin\theta \simeq \theta$.

$$\therefore\ \frac{d^2\theta}{dt^2} = -\frac{g}{L}\theta.$$

$$\therefore\ \theta = \theta_0 \sin(\omega_0 t + \phi).$$

where, $\omega_0 = \left(\frac{g}{L}\right)^{1/2}$.

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{g}{L}}.$$

Maximum value of amplitude is θ_0.

At $t = 0$, $\theta = \theta_0 \sin\phi$.

& $\frac{d\theta}{dt} = \omega_0 \theta_0 \sin\phi$.

How large can θ_0 be so that assumption $\sin\theta = \theta$ is valid?

Answer is relative!

θ (deg.)	(Period $/ 2\pi\sqrt{L/g}$).
0	1.0000
5	1.0005
10	1.0019
15	1.0043
20	1.0077
30	1.0174
45	1.0396
60	1.0719

Can we deal with large amplitudes analytically?

Anharmonic oscillator

Let $\sin\theta = \theta - \frac{\theta^3}{6}$. (restrict to $\mathcal{O}(\theta^3)$.)

$$\therefore \boxed{\frac{d^2\theta}{dt^2} + \omega_0^2\theta - \frac{\omega_0^2}{6}\theta^3 = 0.}$$

$$\omega_0^2 = \frac{g}{L}.$$

Assume an approximate solⁿ.

$$\theta = \theta_0 \sin\omega t + \epsilon\,\theta_0 \sin 3\omega t.$$

ϵ is dimensionless constant

$\epsilon \ll 1$ for $\theta_0 \ll 1$.

Why $3\omega t$?

$$\sin^3 x = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x.$$

Thus θ^3 term in the diffl. eqⁿ. will generate from the cube of $\sin\omega t$ a term in $\sin 3\omega t$.

$$\boxed{\omega, \epsilon = ?}$$

Note: $\epsilon \sin 3\omega t$ in trial solⁿ. will generate a term $\sim \sin 9\omega t$!

$$\ddot{\theta} = -\omega^2\theta_0 \sin\omega t - 9\omega^2\epsilon\theta_0 \sin 3\omega t.$$

$$\theta^3 = \theta_0^3(\sin^3\omega t + 3\epsilon \sin^2\omega t \sin 3\omega t + \dots)$$

use $\sin^3 x \equiv \frac{3}{4}\sin x - \frac{1}{4}\sin 3x$.

discarding terms $O(\epsilon^2)$ & $O(\epsilon^3)$.

$$\ddot{\theta} = -\omega^2\theta_0 \sin\omega t - 9\omega^2\epsilon\theta_0 \sin 3\omega t$$

$$\omega_0^2\theta = +\omega_0^2\theta_0 \sin\omega t + \omega_0^2\epsilon \sin 3\omega t.$$

$$-\frac{1}{6}\omega_0^2\theta^3 = -\frac{3}{24}\omega_0^2\theta_0^3 \sin\omega t + \frac{\omega_0^2\theta_0^3}{24}\sin 3\omega t.$$

$$-\frac{\omega_0^2}{2}\theta_0^3\epsilon \sin^2\omega t \sin 3\omega t.$$

coeff. of $\sin\omega t$ term must add up to zero,

$$\Rightarrow -\omega^2 + \omega_0^2 - \frac{3}{24}\omega_0^2\theta_0^2 = 0.$$

$$\Rightarrow \omega^2 = \omega_0^2\left(1 - \frac{1}{8}\theta_0^2\right).$$

$$\therefore \omega = \omega_0\sqrt{1 - \frac{1}{8}\theta_0^2}.$$

$$\approx \omega_0\left(1 - \frac{1}{16}\theta_0^2\right).$$

e.g., for $\theta_0 = 0.3$ rad.,

$$\frac{\Delta\omega}{\omega} \approx -10^{-2}.$$

$$\Delta\omega = \omega - \omega_0.$$

$\therefore$ Freq. of pendulum depends on amplitude for large amplitudes.

coeff. of $\sin 3\omega t$ term upto $\mathcal{O}(\theta_0^2)$,

$$-9\omega^2\epsilon + \omega_0^2\epsilon + \frac{\omega_0^2}{24}\theta_0^2 = 0.$$

upto $\mathcal{O}(\theta_0^2)$,

$$\omega^2 \approx \omega_0^2.$$

$$\boxed{\therefore \ \epsilon \approx \frac{\theta_0^2}{192}}$$

For $\theta_0 = 0.3$ rad.,

we have, $\epsilon \approx 10^{-3}$.

Analogy

Harmonic oscillator & LC circuit

Voltage across the capacitance C is,

$$V_C = \frac{Q}{C}. \quad \left(\text{where } Q \text{ is the charge on the capacitance}\right)$$

Current in the ckt. in series with capacitance is,

$$I = -\frac{dQ}{dt} \quad \text{or,} \quad Q = -\int I\,dt.$$

The voltage across the inductor L is,

$$V_L = -L\frac{dI}{dt}.$$

In LC ckt.,

$$-L\frac{dI}{dt} + \frac{Q}{C} = 0 = L\frac{d^2Q}{dt^2} + \frac{Q}{C}.$$

$Q \leftrightarrow x$

$L \leftrightarrow M$

$\frac{1}{C} \leftrightarrow C_{spring}$.

$Q = Q_0 \sin(\omega_0 t + \phi)$

$\omega_0 = \left(\frac{1}{LC}\right)^{1/2}$.