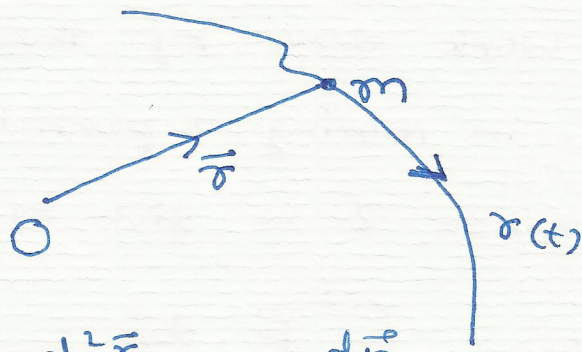


Mechanics of a particle



$$\vec{F} = m \frac{d^2 \vec{r}}{dt^2} = \frac{d\vec{p}}{dt},$$

where \vec{F} is the force acting on m .

Soln. of the eq. is of the form $\vec{r} = \vec{r}(t)$.

(Determines the trajectory of mass m)

If no external force is acting,

$$\vec{F} = \frac{d\vec{p}}{dt} = 0$$

$$\Rightarrow \vec{p} = m\vec{v} = \text{const.}$$

Angular momentum about point O is defined as,

$$\vec{L} = \vec{r} \times \vec{p}$$

$$\therefore \dot{\vec{L}} = \frac{d\vec{L}}{dt} = \frac{d}{dt} (\vec{r} \times \vec{p}) = \dot{\vec{r}} \times \vec{p} + \vec{r} \times \dot{\vec{p}}.$$

$$\text{But } \dot{\vec{r}} \times \vec{p} = \dot{\vec{r}} \times m\dot{\vec{r}} = 0.$$

$$\& \vec{r} \times \dot{\vec{p}} = \vec{r} \times \vec{F}.$$

$$\therefore \boxed{\dot{\vec{L}} = \frac{d\vec{L}}{dt} = \vec{r} \times \vec{F}}$$

If $\vec{F} = 0$, $\dot{\vec{L}} = 0$, $\vec{L} = \text{a constant}$.

(e.g., planets moving around sun).

The work done by the total external force in moving a particle from position 1 to position 2 is given by,

$$\begin{aligned}
 W_{12} &= \int_1^2 \vec{F} \cdot d\vec{r} \\
 &= \int_1^2 m \frac{d\vec{v}}{dt} \cdot d\vec{r} \quad \text{if } m \text{ is constant} \\
 &= \int_1^2 m \frac{d\vec{v}}{dt} \cdot \left(\frac{d\vec{r}}{dt}\right) dt \\
 &= \frac{1}{2} m \int_1^2 d(v^2) \\
 &= \frac{1}{2} m (v_2^2 - v_1^2) \\
 &= T_2 - T_1 \quad (\text{kinetic energies of the particle at positions } 2 \text{ \& } 1.)
 \end{aligned}$$

If $T_1 > T_2$, $W_{12} < 0$

Work is done by the particle against the force & its k.E. has reduced.

If $T_1 < T_2$,

work is done by the force on the particle & its k.E. increases.

If the force field is such that work done along a closed path is 0 then the force is said to be "conservative".

i.e., if $\oint \vec{F} \cdot d\vec{r} = 0$, then \vec{F} is conservative.

$$\oint \vec{F} \cdot d\vec{r} = \int_{\sigma} (\nabla \times \vec{F}) \cdot d\vec{\sigma}.$$

\therefore for a conservative force field,

$$\nabla \times \vec{F} = 0.$$

$$(\because \nabla \times (\nabla \phi) = 0.)$$

$$\Rightarrow \vec{F} = -\nabla \phi.$$

↓
potential energy of the particle.

$$\therefore W_{12} = \int_1^2 \vec{F} \cdot d\vec{r} = - \int_1^2 \nabla \phi \cdot d\vec{r}.$$

$$= - \int_1^2 \left(\frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz \right).$$

$$= - \int_1^2 d\phi = \phi_1 - \phi_2.$$

$$\text{But } W_{12} = T_2 - T_1 \Rightarrow T_2 - T_1 = \phi_1 - \phi_2.$$

$$\therefore U = T_1 + \phi_1 = T_2 + \phi_2 = \text{const.}$$

Useful Identities

$$\vec{\nabla} = \hat{e}_1 V_1 + \hat{e}_2 V_2 + \hat{e}_3 V_3.$$

$$= \frac{\hat{e}_1}{h_2 h_3} (h_2 h_3 V_1) + \frac{\hat{e}_2}{h_1 h_3} (h_1 h_3 V_2)$$

$$\nabla \phi = \frac{\hat{e}_1}{h_1} \frac{\partial}{\partial u_1} + \frac{\hat{e}_2}{h_2} \frac{\partial}{\partial u_2} + \frac{\hat{e}_3}{h_3} \frac{\partial}{\partial u_3} + \frac{\hat{e}_3}{h_1 h_2} (h_1 h_2 V_3).$$

$$\nabla \cdot \vec{\nabla} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} (h_2 h_3 V_1) + \frac{\partial}{\partial u_2} (h_3 h_1 V_2) + \frac{\partial}{\partial u_3} (h_1 h_2 V_3) \right].$$

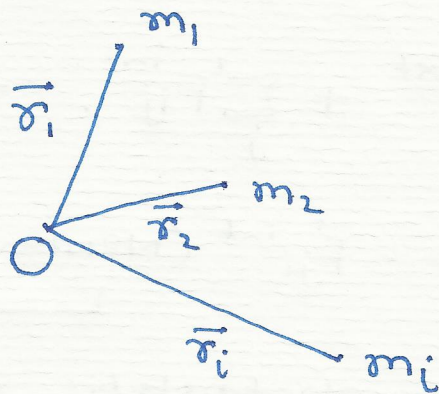
$$\nabla \times \vec{\nabla} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{e}_1 & h_2 \hat{e}_2 & h_3 \hat{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 V_1 & h_2 V_2 & h_3 V_3 \end{vmatrix}$$

$$\nabla^2 \phi = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} \left(\frac{h_1 h_2 h_3}{h_1} \right) \frac{\partial}{\partial u_1} + \dots \right].$$

cyclic term.

Mechanics of a system of particles

System of N particles



Let \vec{F}^{ext} be the total external force acting on the system of N particles with \vec{F}_i^{ext} being the external force acting on i^{th} particle. Also, internal force $\sum_j \vec{F}_{ij}^{\text{int}}$ acts on i^{th} particle due to all the other particles. i.e., $\vec{F}_i^{\text{int}} = \sum_j' \vec{F}_{ij}^{\text{int}}$ (prime \equiv excludes i^{th} particle)

Total mass of the system is given by $M = \sum_{i=1}^N m_i \equiv \sum_i m_i$. — (ii)

The center of mass of the system is defined as the point whose position vector is given by,

$$\vec{R}_{\text{cm}} = \frac{\sum_i m_i \vec{r}_i}{\sum_i m_i} = \frac{1}{M} \sum_i m_i \vec{r}_i. \quad \text{— (iii)}$$

Let

The equation of motion (EOM) of the i^{th} particle can be written as,

$$\vec{p}_i = \vec{F}_i^{\text{ext}} + \sum_j' \vec{F}_{ij}^{\text{int}} \quad \text{--- (iv)}$$

Now, $\vec{F}_{ij}^{\text{int}} = -\vec{F}_{ji}^{\text{int}}$ (Newton's 3rd law).
└── (v)

(examples: electrostatic, gravitational)
 Note: EM force is a velocity dep. force & does not act along the line joining i^{th} & j^{th} particles.

∴ EOM of the system is

$$\sum_i \dot{\vec{p}}_i = \sum_i \vec{F}_i^{\text{ext}} + \sum_{ij}' \vec{F}_{ij}^{\text{int}} \quad \text{--- (vi)}$$

or, $\frac{d^2}{dt^2} \sum_i m_i \vec{r}_i = \sum_i \vec{F}_i^{\text{ext}} + \sum_{ij}' \vec{F}_{ij}^{\text{int}}$
└── (vii)

Consider $\sum_{ij}' \vec{F}_{ij}^{\text{int}}$.

i & j are dummy indices,

$$\Rightarrow \sum_{ij}' \vec{F}_{ij}^{\text{int}} = \sum_{ij}' \vec{F}_{ji}^{\text{int}} \quad \text{--- (viii)}$$

Further, using (v) & (viii),

$$\begin{aligned} \sum_{ij}' \vec{F}_{ij}^{\text{int}} &= \frac{1}{2} \left(\sum_{ij}' \vec{F}_{ij}^{\text{int}} + \sum_{ij}' \vec{F}_{ji}^{\text{int}} \right) \\ &= \frac{1}{2} \sum_{ij}' (\vec{F}_{ij}^{\text{int}} + \vec{F}_{ji}^{\text{int}}) = 0 \quad \text{--- (ix)} \end{aligned}$$

$$\therefore \frac{d^2}{dt^2} \sum_i m_i \vec{r}_i = \sum_i \vec{F}_i^{\text{ext}} = \vec{F}^{\text{ext}}$$

$$\Rightarrow \frac{d^2}{dt^2} (M \vec{R}_{\text{cm}}) = \vec{F}^{\text{ext}}$$

$$\text{i.e., } M \frac{d^2}{dt^2} \vec{R}_{\text{cm}} = \vec{F}^{\text{ext}} \quad \text{--- (x)}$$

($\because \frac{dM}{dt} = 0$.)

$$\text{i.e., } \boxed{M \ddot{\vec{R}}_{\text{cm}} = \vec{F}^{\text{ext}}} \quad \text{--- (xi)}$$

Also, total linear momentum of the system is given by,

$$\begin{aligned} \vec{P} &= \sum_i m_i \dot{\vec{r}}_i = \frac{d}{dt} \sum_i m_i \vec{r}_i \\ &= \frac{d}{dt} (M \vec{R}_{\text{cm}}) = M \dot{\vec{R}}_{\text{cm}} \end{aligned}$$

--- (xii)

Using (xi) & (xii),

$$\boxed{\dot{\vec{P}} = \vec{F}^{\text{ext}}} \quad \text{--- (xiii)}$$

Center of mass behaves like a particle whose mass is equal to the total mass of the system & is acted upon by total external force \vec{F}^{ext} .

Corrolaries

(1) This exist only for force fields for which $\vec{F}_{ij}^{\text{int}} = \vec{F}_{ji}^{\text{int}}$.

(2) If total external force $\vec{F}^{\text{ext}} = 0$, then the total linear momentum of the system of N particles is conserved.

i.e., if $\vec{F}^{\text{ext}} = 0$, then $\dot{\vec{P}} = 0$ or, $\vec{P} = \text{const.}$

Angular momentum of the system of particles

$$\vec{l}_i = \vec{r}_i \times \vec{p}_i \quad \text{angular momentum}$$

of the i^{th} particle about some point.

\therefore total angular momentum of the system

$$\vec{L} = \sum_i \vec{l}_i = \sum_i \vec{r}_i \times \vec{p}_i.$$

\therefore Torque acting on the system is,

$$\vec{\tau} = \frac{d\vec{L}}{dt} = \frac{d}{dt} \sum_i \vec{r}_i \times \vec{p}_i.$$

$$= \sum_i \dot{\vec{r}}_i \times \vec{p}_i + \sum_i \vec{r}_i \times \dot{\vec{p}}_i.$$

$$\text{But, } \sum_i \dot{\vec{r}}_i \times \vec{p}_i = \sum_i \dot{\vec{r}}_i \times m_i \dot{\vec{r}}_i \\ = \sum_i m_i \dot{\vec{r}}_i \times \dot{\vec{r}}_i = 0.$$

$$\text{Also, } \sum_i \vec{r}_i \times \dot{\vec{p}}_i = \sum_i \vec{r}_i \times \left(\vec{F}_i^{\text{ext}} + \sum_j' \vec{F}_{ij}^{\text{int}} \right). \\ \therefore = \sum_i \vec{r}_i \times \vec{F}_i^{\text{ext}}.$$

Because,

$$\sum_i \sum_j' \vec{r}_i \times \vec{F}_{ij}^{\text{int}} = \sum_j \sum_i' \vec{r}_j \times \vec{F}_{ji}^{\text{int}}.$$

$$\therefore \sum_{ij}' \vec{r}_i \times \vec{F}_{ij}^{\text{int}} = \frac{1}{2} \left(\sum_{ij}' \left\{ \vec{r}_i \times \vec{F}_{ij}^{\text{int}} + \vec{r}_j \times \vec{F}_{ji}^{\text{int}} \right\} \right).$$

$$= \frac{1}{2} \sum_{ij}' (\vec{r}_i - \vec{r}_j) \times \vec{F}_{ij}^{\text{int}}.$$

But $\vec{F}_{ij}^{\text{int}}$ acts along $\vec{r}_i - \vec{r}_j$
for case under consideration.

$$\therefore \sum_{ij}' \vec{r}_i \times \vec{F}_{ij}^{\text{int}} = 0.$$

$$\therefore \vec{\tau} = \frac{d\vec{L}}{dt} = \sum_i \vec{r}_i \times \vec{F}_i^{\text{ext}} = \sum_i \vec{\tau}_i.$$

Total torque on the system is equal to the vector sum of the torques acting on the individual particles.

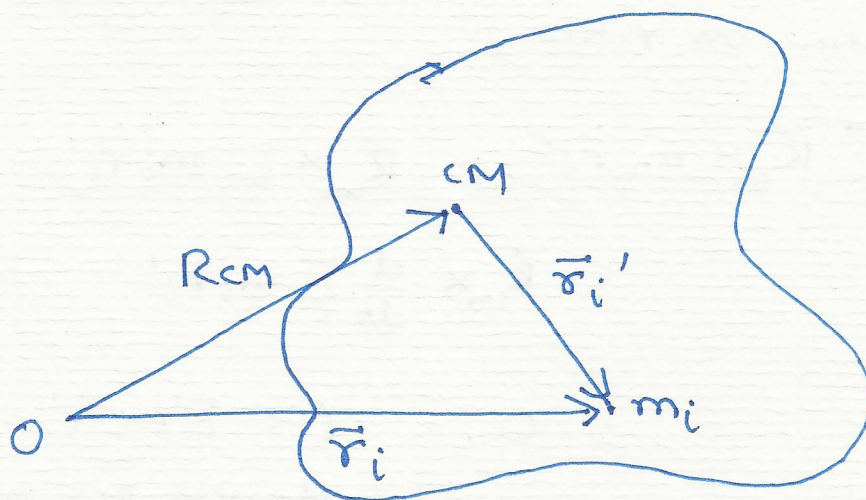
Note!

• Only valid if $\vec{F}_{ij}^{\text{int}}$ acts along $(\vec{r}_i - \vec{r}_j)$.

• If $\vec{\tau} = 0 \Rightarrow \frac{d\vec{L}}{dt} = 0$.

i.e., $\vec{L} = \text{const.}$

i.e., if total external torque acting on the system of N particles is zero, then the total angular momentum of the system is conserved.



m_i & \vec{r}_i denote the mass and the position vector of the i^{th} particle with reference to point O .

Let \vec{r}_i' be the position vector of the i^{th} particle with reference to the center of mass.

$$\therefore \vec{r}_i = \vec{R}_{cm} + \vec{r}_i'$$

$$\therefore \dot{\vec{r}}_i = \dot{\vec{R}}_{cm} + \dot{\vec{r}}_i'$$

\Rightarrow Total angular momentum of the system of particles is,

$$\vec{L} = \sum_i \vec{r}_i \times \vec{p}_i$$

$$= \sum_i (\vec{R}_{cm} + \vec{r}_i') \times m_i (\dot{\vec{R}}_{cm} + \dot{\vec{r}}_i')$$

$$= \sum_i m_i \vec{R}_{cm} \times \dot{\vec{R}}_{cm} + \sum_i \vec{R}_{cm} \times m_i \dot{\vec{r}}_i' + \sum_i m_i \vec{r}_i' \times \dot{\vec{R}}_{cm} + \sum_i m_i \vec{r}_i' \times \dot{\vec{r}}_i'$$

2nd term on r.h.s.,

$$\begin{aligned}\sum_i \vec{R}_{cm} \times m_i \dot{\vec{r}}_i' &= \vec{R}_{cm} \times \sum_i m_i \dot{\vec{r}}_i' \\ &= \vec{R}_{cm} \times \frac{d}{dt} \sum_i m_i \vec{r}_i'.\end{aligned}$$

$$= 0. \quad (\because \sum m_i \vec{r}_i' = 0.)$$

Distances are measured
w.r.t. CM.

3rd term on r.h.s.,

$$\sum_i m_i \vec{r}_i' \times \dot{\vec{R}}_{cm} = \left(\sum_i m_i \vec{r}_i' \right) \times \dot{\vec{R}}_{cm}.$$

$$= 0. \quad (\text{as above}).$$

$$\therefore \vec{L} = \sum_i m_i \vec{R}_{cm} \times \dot{\vec{R}}_{cm} + \sum_i m_i \vec{r}_i' \times \dot{\vec{r}}_i'.$$

$$= \vec{R}_{cm} \times \dot{\vec{R}}_{cm} \sum_i m_i + \sum_i \vec{r}_i' \times m_i \dot{\vec{r}}_i'.$$

$$= \vec{R}_{cm} \times \dot{\vec{R}}_{cm} M + \sum_i \vec{r}_i' \times \vec{p}_i'.$$

$$\text{i.e., } \vec{L} = \underbrace{(\vec{R}_{cm} \times \vec{P}_{cm})}_{\text{ang. mom. of the total mass concentrated at the center of mass about point O.}} + \underbrace{\vec{L}'}_{\text{ang. mom. of the system about its center of mass.}}$$

ang. mom. of the
total mass concentrated
at the center of mass
about point O.

ang. mom. of the
system about its
center of mass.