Quantum Harmonic Oscillator

PH103 (Physics-I)
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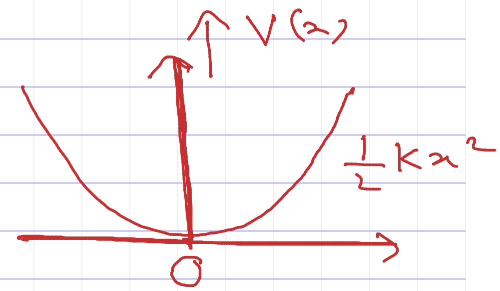
$$\hat{H} = \frac{\hat{p}_x^2}{2m} + \frac{1}{2} m \omega_0^2 \hat{x}^2$$



$k = m\omega_0^2$.

Let $\hat{P} = \frac{1}{p_0}\hat{p}_x$, $\hat{Q} = \frac{1}{x_0}\hat{x}$ } Dimensionless version

$p_0 = m^{\alpha}\omega^{\beta}\hbar^{\gamma}$ s.t., $[p_0] = MLT^{-1}$

$x_0 = m^{a}\omega^{b}\hbar^{c}$ s.t., $[x_0] = M^0 L T^0$

$[\omega_0] = T^{-1}$
$[\hbar] = ML^2T^{-1}$
$[m] = M$

$\Downarrow$

$$x_0 = \sqrt{\frac{\hbar}{m\omega_0}}, \quad p_0 = \sqrt{m\omega_0\hbar}$$

$$[\hat{Q}, \hat{P}] = \frac{1}{x_0 p_0}[x, p_x] = i. \quad \text{—— (i)} \quad (\because [x, p_x] = i\hbar)$$

$$\Rightarrow \hat{H} = \frac{p_0^2}{2m}\hat{P}^2 + \frac{1}{2}m\omega_0^2 x_0^2 \hat{Q}^2 = \frac{\hbar\omega_0}{2}(\hat{P}^2 + \hat{Q}^2) \quad \text{—— (ii)}$$

Let, $\hat{a} = \frac{1}{\sqrt{2}}(\hat{Q} + i\hat{P})$, $\hat{a}^{+} = \frac{1}{\sqrt{2}}(\hat{Q} - i\hat{P})$

$$\Rightarrow [\hat{a}, \hat{a}^{+}] = \frac{1}{2}[\hat{Q} + i\hat{P}, \hat{Q} - i\hat{P}]$$

$$= \frac{1}{2}\left\{-i[\hat{Q}, \hat{P}] + i[\hat{P}, \hat{Q}]\right\}$$

$$= 1 \quad (\text{using (i).})$$

$$\Rightarrow [\hat{a}, \hat{a}^{+}] = 1 \quad \text{—— (iii)}$$

$$\therefore \hat{H} = \hbar\omega_0\left\{\left(\frac{\hat{Q} - i\hat{P}}{\sqrt{2}}\right)\left(\frac{\hat{Q} + i\hat{P}}{\sqrt{2}}\right) - i[\hat{Q}, \hat{P}]\right\} = \hbar\omega_0\left(\hat{a}^{+}\hat{a} + \frac{1}{2}\right) \quad \text{—— (iv)}$$

Define a state $|n\rangle$ and a number operator $\hat{N} = \hat{a}^{+}\hat{a}$ (v), s.t., $\hat{N}|n\rangle = n|n\rangle$. (vi)

n → number of particles in state $|n\rangle$.

$$\Rightarrow [\hat{N}, a^{+}] = [\hat{a}^{+}\hat{a}, \hat{a}^{+}]$$
$$= \hat{a}^{+}[\hat{a}, \hat{a}^{+}] + [\hat{a}^{+}, \hat{a}^{+}]\hat{a}$$
$$= \hat{a}^{+}. \qquad \therefore [\hat{N}, \hat{a}^{+}] = \hat{a}^{+}. \quad \text{—— (vii)}$$

using (iii)

Similarly,

$$[\hat{N}, \hat{a}] = [\hat{a}^{+}\hat{a}, \hat{a}]$$
$$= \hat{a}^{+}[\hat{a}, \hat{a}] + [\hat{a}^{+}, \hat{a}]\hat{a}$$
$$= -\hat{a} \qquad \therefore [\hat{N}, \hat{a}] = -\hat{a}. \quad \text{—— (viii)}$$

using (iii)

Now, $[\hat{N}, \hat{a}^{+}]|n\rangle = a^{+}|n\rangle$.

$$\Rightarrow (\hat{N}\hat{a}^{+} - \hat{a}^{+}\hat{N})|n\rangle = a^{+}|n\rangle.$$

$(|n\rangle)^{+} = \langle n|$.

$$\Rightarrow \hat{N}\hat{a}^{+}|n\rangle - n\hat{a}^{+}|n\rangle = \hat{a}^{+}|n\rangle.$$

$(C_n)^{+} = C_n^{*}$

$$\therefore \hat{N}\{\hat{a}^{+}|n\rangle\} = (n+1)\{\hat{a}^{+}|n\rangle\}$$

$$\Rightarrow \hat{a}^{+}|n\rangle = C_n|n+1\rangle. \quad (\text{action of } \hat{N})$$

$$\therefore (\hat{a}^{+}|n\rangle)^{+}\hat{a}^{+}|n\rangle = (C_n|n+1\rangle)^{+}C_n|n+1\rangle.$$

$$\Rightarrow \langle n|\hat{a}\hat{a}^{+}|n\rangle = |C_n|^2\langle n+1|n+1\rangle.$$

$$\Rightarrow \langle n|1 + \hat{a}^{+}\hat{a}|n\rangle = |C_n|^2$$

$$(n+1) = |C_n|^2. \quad \Rightarrow C_n = \sqrt{n+1}.$$

$$\therefore \hat{a}^{+}|n\rangle = \sqrt{n+1}\,|n+1\rangle. \quad \text{—— (ix)}$$

Similarly, $[\hat{N}, \hat{a}]|n\rangle = -\hat{a}|n\rangle$.

$\Rightarrow (\hat{N}\hat{a} - \hat{a}\hat{N})|n\rangle = -\hat{a}|n\rangle$.

$\therefore \hat{N}\{\hat{a}|n\rangle\} = (n-1)\{\hat{a}|n\rangle\}$

$\Rightarrow \hat{a}|n\rangle = d_n|n-1\rangle$ (action of $\hat{N}$)

$\therefore (\hat{a}|n\rangle)^{\dagger}\hat{a}|n\rangle = (d_n|n-1\rangle)^{\dagger} d_n|n-1\rangle$.

$\Rightarrow \langle n|\hat{a}^{\dagger}\hat{a}|n\rangle = \langle n-1|d_n^* d_n|n-1\rangle$.

$\therefore n\langle n|n\rangle = |d_n|^2\langle n-1|n-1\rangle$.

$\Rightarrow d_n = \sqrt{n}$.

$\therefore \hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle$ —— (x)

Eq. (ix) and (x) signifies that:

$\hat{a}$ is a lowering (destruction) operator.

& $\hat{a}^{\dagger}$ is a raising (creation) operator.

Eq. (iv) $\Rightarrow \hat{H} = \hbar\omega_0(\hat{a}^{\dagger}\hat{a} + \frac{1}{2})$

$\therefore \hat{H}|n\rangle = \hbar\omega_0(\hat{a}^{\dagger}\hat{a} + \frac{1}{2})|n\rangle$,

$= (n+\frac{1}{2})\hbar\omega_0|n\rangle$.

$\Rightarrow E_n = (n+\frac{1}{2})\hbar\omega_0$ corresponding to the eigenstate $|n\rangle$.

Also, $\hat{a}|0\rangle = \sqrt{0}|0\rangle = 0$. $\therefore |0\rangle$ is lowest state.

$$\hat{a}^{+} = \frac{1}{\sqrt{2}}(\hat{Q} - i\hat{P}), \quad \hat{a} = \frac{1}{\sqrt{2}}(\hat{Q} + i\hat{P}).$$

Now, $x_0 = \sqrt{\frac{\hbar}{m\omega_0}}$, $p_0 = \sqrt{m\omega_0\hbar}$

$$\therefore \quad \hat{x} = x_0 \hat{Q} = \frac{x_0}{\sqrt{2}}(\hat{a} + \hat{a}^{+}) = \sqrt{\frac{\hbar}{2m\omega_0}}(\hat{a} + \hat{a}^{+}).$$

$$\hat{p}_x = p_0 \hat{P} = \frac{p_0}{\sqrt{2}\,i}(\hat{a} - \hat{a}^{+}) = \frac{1}{i}\sqrt{\frac{m\omega_0\hbar}{2}}(\hat{a} - \hat{a}^{+}).$$

In ground state $|0\rangle$,

$$\therefore \quad \langle\hat{x}\rangle = \langle 0|\sqrt{\frac{\hbar}{2m\omega_0}}(\hat{a} + \hat{a}^{+})|0\rangle = 0.$$

$$\langle\hat{p}_x\rangle = \langle 0|\frac{1}{i}\sqrt{\frac{m\omega_0\hbar}{2}}(a - a^{+})|0\rangle = 0$$

$$\langle\hat{x}^2\rangle = \langle 0|\frac{\hbar}{2m\omega_0}(\hat{a}\hat{a} + \hat{a}\hat{a}^{+} + \hat{a}^{+}\hat{a} + \hat{a}^{+}\hat{a}^{+})|0\rangle$$

$$= \langle 0|\frac{\hbar}{2m\omega_0}\hat{a}\hat{a}^{+}|0\rangle$$

(other terms do not lead to a non-zero contribution)

$$= \frac{\hbar}{2m\omega_0}.$$

$$\langle p_x^2\rangle = \langle 0| -\frac{m\omega_0\hbar}{2}(\hat{a}\hat{a} - \hat{a}\hat{a}^{+} - \hat{a}^{+}\hat{a} + \hat{a}^{+}\hat{a}^{+})|0\rangle$$

$$= \langle 0|\frac{m\omega_0\hbar}{2} a a^{+}|0\rangle$$

$$= \frac{m\omega_0\hbar}{2}.$$

Now by definition Δx is root mean squared deviation, i.e., $\Delta x = \sqrt{\langle(x - \langle x\rangle)^2\rangle}$

$$= \sqrt{\langle x^2 - 2x\langle x\rangle + \langle x\rangle^2\rangle}$$

$$= \sqrt{\langle x^2\rangle - 2\langle x\rangle^2 + \langle x\rangle^2} = \sqrt{\langle x^2\rangle - \langle x\rangle^2}.$$

Similarly, $\Delta p_x = \sqrt{\langle p_x^2 \rangle - \langle p_x \rangle^2}$

$\therefore$ In the ground state, the uncertainty product,

$$\Delta x \, \Delta p_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}\sqrt{\langle p_x^2 \rangle - \langle p_x \rangle^2}$$

$$= \sqrt{\frac{\hbar}{2m\omega_0}\,\frac{m\omega_0\hbar}{2}}$$

$$= \frac{\hbar}{2}.$$

(i.e., $|0\rangle$ has a minimum uncertainty product).

Similarly, it can be easily shown that in state $|n\rangle$, $\Delta x \Delta p_x = (2n+1)\frac{\hbar}{2}$.