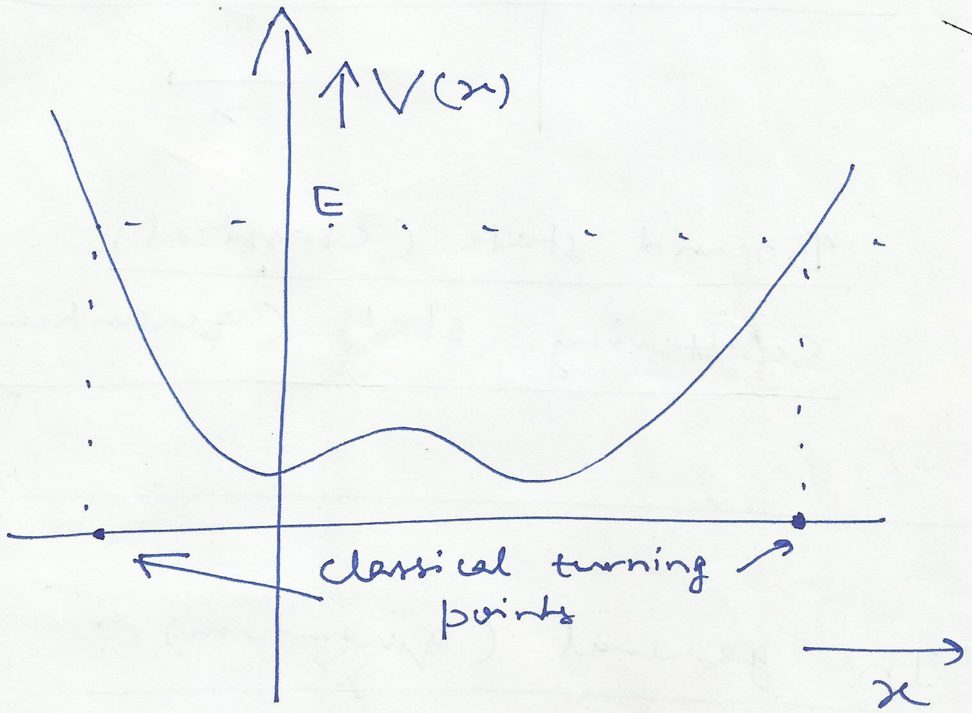
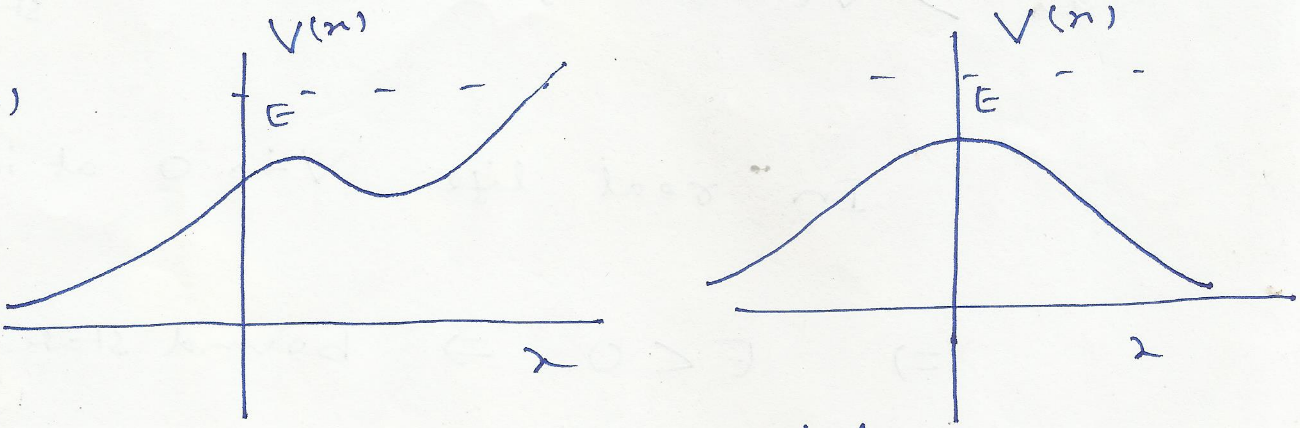
Bound States & Scattering States

(I.)



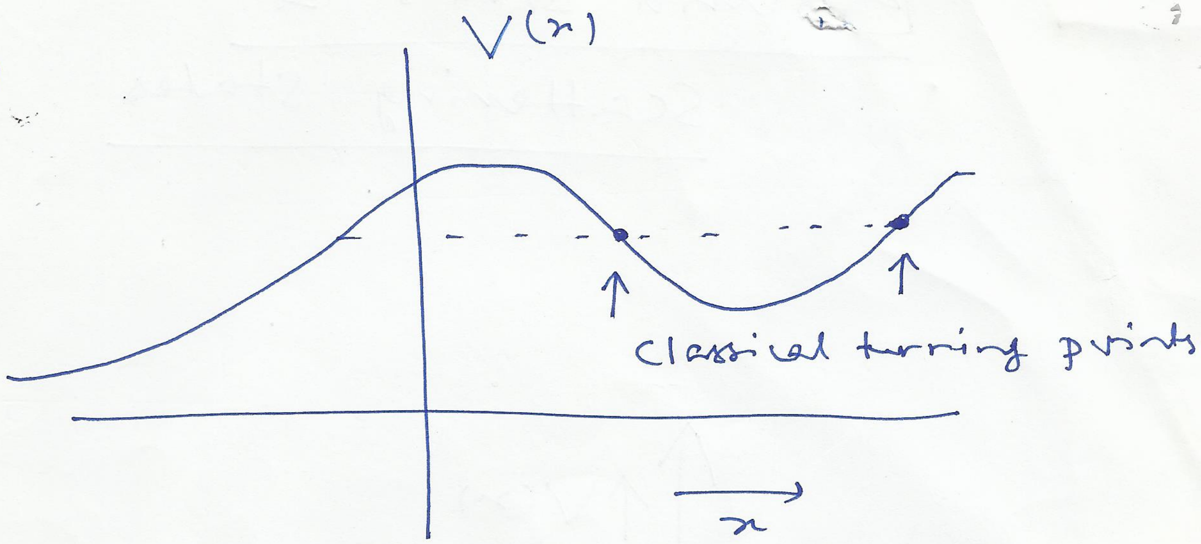
Bound state

(II.)



Scattering states

(III.)



Bound state (classical)

Scattering state (quantum)

In general (quantum) :

- $E < V(-\infty)$ & $V(+\infty)$ $\Rightarrow$ Bound state
- $E > V(-\infty)$ or, $V(+\infty)$ $\Rightarrow$ Scattering state

In real life $V \to 0$ at infinity.

$\Rightarrow$ $E < 0$ $\Rightarrow$ bound state.

& $E > 0$ $\Rightarrow$ scattering state.

Delta function ('more precisely' delta distribution*) :

$$\delta(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases}$$

$$\text{s.t.,} \quad \int_{-\infty}^{\infty} \delta(x)\, dx = 1.$$

Also, $f(x)\,\delta(x-a) = f(a)\,\delta(x-a)$.

($\because$ the product is zero except at $x = a$).

$$\int_{-\infty}^{\infty} f(x)\,\delta(x-a)\,dx = f(a)\int_{-\infty}^{\infty}\delta(x-a)\,dx = f(a).$$

* also called generalized function.

⟹ Schrödinger time independent equation for $V(x) = -\alpha\,\delta(x)$, $(\alpha > 0)$.

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - \alpha\,\delta(x)\,\psi = E\psi .$$

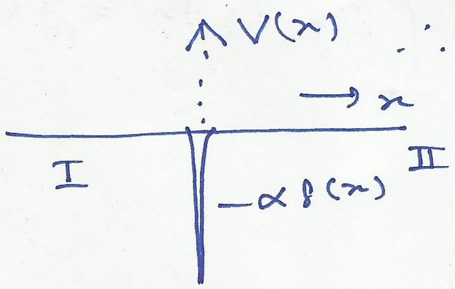
This has both:

bound states $(E < 0)$.

& scattering states $(E > 0)$.

⟹ Bound state solutions for $V(x) = -\alpha\,\delta(x)$, $\alpha > 0$.

For $x < 0$, $V(x) = 0$



$$\therefore \frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi = K^2\psi .$$

$$K = \sqrt{-\frac{2mE}{\hbar^2}} .$$

$(\because E < 0.)$

The general solution is

$$\psi_I(x) = A e^{-Kx} + B e^{Kx} .$$

But first term blows up at $x \to -\infty$.

$\therefore A = 0$ (choose).

$$\Rightarrow \psi_I = B e^{Kx} . \quad (x < 0).$$

Similarly for region II,

$$\psi_{II}(x) = F e^{-Kx} \quad (x>0).$$

$(\because$ as $x \to +\infty$, $\psi_{II} \to 0.)$.

Boundary condition ψ is continuous **

(even for a piecewise discontinuous function).

$\Rightarrow B = F$.

$$\Rightarrow \quad \psi(x) = \begin{cases} B e^{Kx} & x \leq 0 \\ B e^{-Kx} & x \geqslant 0 \end{cases}$$

Integrate schrödinger eq. "suitably":

$$\lim_{\epsilon \to 0} \quad -\frac{\hbar^2}{2m} \int_{-\epsilon}^{+\epsilon} \frac{d^2\psi}{dx^2}\, dx + \int_{-\epsilon}^{\epsilon} V(x)\,\psi(x)\, dx = \underbrace{E \int_{-\epsilon}^{\epsilon} \psi(x)\, dx}_{\to 0}.$$

$$\Rightarrow \quad \Delta\left(\frac{d\psi}{dx}\right) = \left.\frac{\partial \psi}{\partial x}\right|_{+\epsilon} - \left.\frac{\partial \psi}{\partial x}\right|_{-\epsilon} = \frac{2m}{\hbar^2} \lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} V(x)\,\psi(x)\, dx$$

$$= -\frac{2m\alpha}{\hbar^2}\,\psi(0).$$

[** $\frac{d\psi}{dx}$ is continuous except at points where $V \to \pm\infty$. It's not useful here.

$$\Rightarrow \quad -2BK = -\frac{2m\alpha}{\hbar^2} B.$$

$$\Rightarrow \quad K = \frac{m\alpha}{\hbar^2}.$$

$$\Rightarrow \quad \frac{m\alpha}{\hbar^2} = \sqrt{-\frac{2mE}{\hbar^2}}.$$

$$\therefore \quad E = -\frac{\hbar^2}{2m}\frac{m^2\alpha^2}{\hbar^4} = -\frac{m\alpha^2}{2\hbar^2}.$$

Normalize ψ, $\Rightarrow \int_{-\infty}^{\infty} |\psi(x)|^2 dx = 2|B|^2 \int_0^{\infty} e^{-2Kx} dx$

$$= \frac{|B|^2}{K} = 1.$$

$$\Rightarrow \quad B = \sqrt{K} = \frac{\sqrt{m\alpha}}{\hbar}.$$

$\therefore$ delta-function well, regardless of its strength has exactly "<u>one</u>" bound state s.t.,

$$\psi(x) = \frac{\sqrt{m\alpha}}{\hbar} e^{-m\alpha|x|/\hbar^2}$$

$$E = -\frac{m\alpha^2}{2\hbar^2}.$$

Scattering states for $V(x) = -\alpha\,\delta(x)$

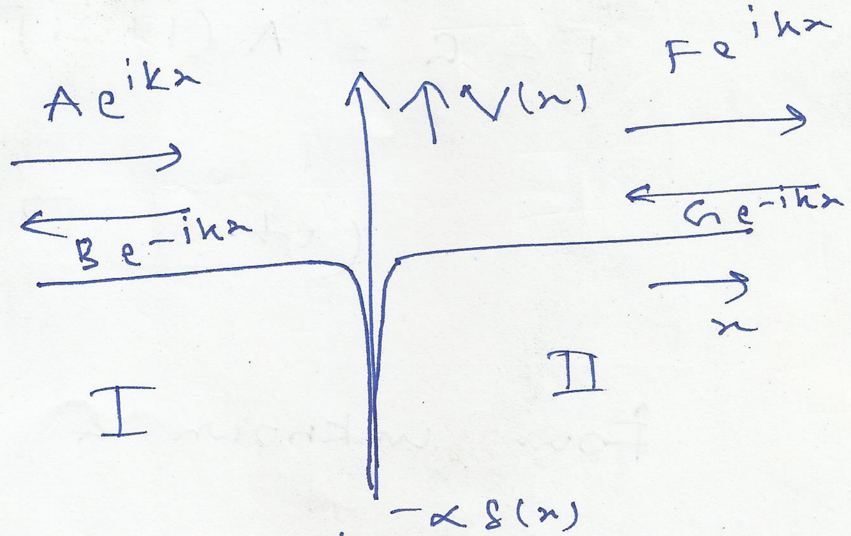
Consider $E > 0$ solutions.

The Schrödinger eqn. reads
(both in region I & II.)

$$\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi = -k^2\psi.$$

s.t., $k = \sqrt{\frac{2mE}{\hbar^2}}$ is real & positive.

$$\psi_I = Ae^{ikx} + Be^{-ikx}.$$



& $\psi_{II} = Fe^{ikx} + Ge^{-ikx}.$

- Continuity at $x=0$

$\Rightarrow$ $F + G = A + B.$

- Discontinuity at $x=0$:

$\frac{d\psi}{dx} = ik(Fe^{ikx} - Ge^{-ikx})$ for $x>0$

$\Rightarrow$ $\left.\frac{d\psi}{dx}\right|_+ = ik(F-G).$

Similarly,

$$\left.\frac{d\psi}{dx}\right|_{-\epsilon} = ik(A-B).$$

$$\Rightarrow \Delta\left(\frac{d\psi}{dx}\right) = ik(F-G-A+B).$$

Also, $\psi(0) = A+B$.

$$\Rightarrow \quad ik(F-G-A+B) = -\frac{2m\alpha}{\hbar^2}(A+B).$$

or,

$$F-G = A(1+2i\Gamma) - B(1-2i\Gamma).$$

$$\left(\text{where, } \Gamma = \frac{m\alpha}{\hbar^2 k}.\right).$$

Four unknown & Two equations

Invoke travelling wave nature of the wave function (with the time-dependent factor $e^{-iEt/\hbar}$) & put physical requirement of particle coming from left & transmitting to right & partially reflecting back.

$\therefore \quad G = 0.$ (nothing to reflect back in region II.)

$\therefore \quad F = A + B.$

& $\quad F = A(1+2i\Gamma) - B(1-2i\Gamma).$

$\left(\& \quad \Gamma = \frac{m\alpha}{\hbar^2 k}. \right).$

$\therefore \quad A + B = A - B + 2i\Gamma A + 2i\Gamma B.$

$\Rightarrow 2B(1-i\Gamma) = 2i\Gamma A.$

$$\therefore \boxed{B = \frac{i\Gamma}{1-i\Gamma} A.}$$

Also, $\boxed{F = \left(\frac{1}{1-i\Gamma}\right) A.}$ $\quad \because \underline{F = A+B.}$

$$\therefore \quad R \equiv \frac{|B|^2}{|A|^2} = \frac{\Gamma^2}{(1+i\Gamma)(1-i\Gamma)}$$

$$= \frac{\Gamma^2}{1+\Gamma^2} = \frac{1}{1+\left(\frac{1}{\Gamma^2}\right)}.$$

$$\therefore \boxed{R = \frac{1}{1 + \frac{\hbar^2 k^4}{m^2 \alpha^2}}}$$

(X)

Also, $T = \dfrac{|F|^2}{|A|^2} = \dfrac{1}{1+\Gamma^2}$

$$\therefore \boxed{T = \dfrac{1}{1 + \dfrac{m^2\alpha^2}{\hbar^4 k^2}}} \qquad \left(\because \Gamma = \dfrac{m\alpha}{\hbar^2 k}\right)$$

Using $E = \dfrac{\hbar^2 k^2}{2m}$,

$$\boxed{\begin{aligned} R &= \dfrac{1}{1 + \left(\dfrac{2\hbar^2}{m\alpha^2}\right) E} \\ T &= \dfrac{1}{1 + \left(\dfrac{m\alpha^2}{2\hbar^2}\right)\dfrac{1}{E}} \end{aligned}}$$

$$\left\{ \begin{aligned} \text{Check} \quad R + T &= \dfrac{1}{1+\Gamma^2} + \dfrac{1}{1+\frac{1}{\Gamma^2}} \\ &= \dfrac{\Gamma^2 + \frac{1}{\Gamma^2} + 2}{\Gamma^2 + \frac{1}{\Gamma^2} + 2} = 1. \end{aligned} \right.$$