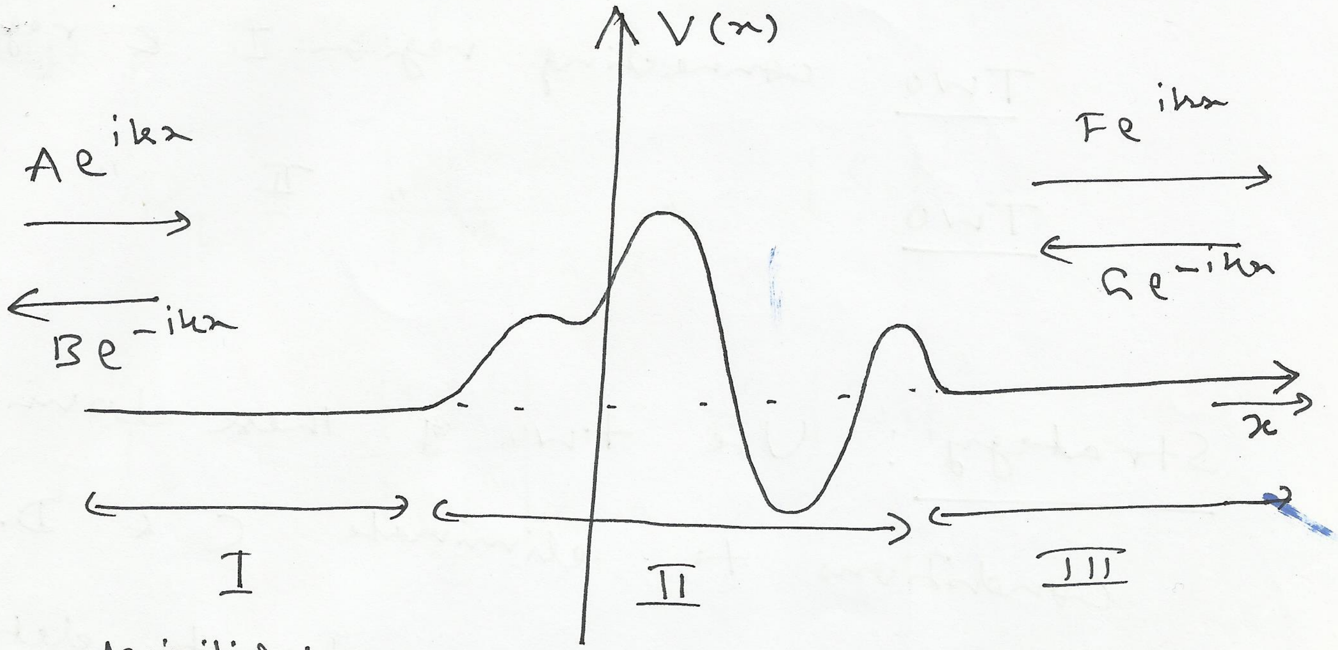
S-matrix



Problem definition:

Scattering from an arbitrary localized potential ($V(x) = 0$ except in region II.).

$V(x) = 0$ in regions I & III.

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi.$$

$$\therefore \frac{d^2\psi}{dx^2} + k^2\psi = 0 \qquad k = \sqrt{\frac{2mE}{\hbar^2}}.$$

Region I: $\psi(x) = Ae^{ikx} + Be^{-ikx}$.

Region III: $\psi(x) = Fe^{ikx} + Ge^{-ikx}$.

In region II, the general solution has the form $\psi(x) = C f(x) + D g(x)$,

here, $f(x)$ & $g(x)$ are the two linearly independent particular solutions.

There are four boundary conditions,

Two connecting region I & region II

Two " " II " " III.

Strategy: Use two of these boundary conditions to eliminate C & D.

Use the other two to determine B & F in terms of A & G.

(outgoing amplitudes) (incoming amplitudes)

i.e.,

$$B = S_{11} A + S_{12} G$$

$$F = S_{21} A + S_{22} G$$

i.e., $$\begin{pmatrix} B \\ F \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} A \\ G \end{pmatrix}.$$

S_{ij} depends on k (& E.).

(iii)

When incoming 'particle' comes from left, $G = 0$

$$\therefore \quad R_l = \left.\frac{|B|^2}{|A|^2}\right|_{G=0} = |S_{11}|^2 .$$

$$T_l = \left.\frac{|F|^2}{|A|^2}\right|_{G=0} = |S_{21}|^2 .$$

When incoming 'particle' comes from right, $A = 0$

$$\therefore \quad R_r = \left.\frac{|F|^2}{|G|^2}\right|_{A=0} = |S_{22}|^2 .$$

$$T_r = \left.\frac{|B|^2}{|G|^2}\right|_{A=0} = |S_{12}|^2 .$$

Help: $S_{i \leftarrow j}$.

x

Transfer matrix M :

Define M via

$$\begin{pmatrix} F \\ G \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} .$$

(Note: M-matrix connect amplitudes to the right of the potential to those to the left of the potential.

(i)

S-matrix for $V(x) = -\alpha\delta(x)$.

For $G = 0$, we obtained

$$\left.\begin{aligned} B &= \frac{i\Gamma}{1-i\Gamma}A \\ F &= \frac{1}{1-i\Gamma}A \end{aligned}\right\} \text{—— (a.)}$$

$$\Gamma = \frac{m\alpha}{\hbar^2 k}.$$

By symmetry, if $A = 0$ s.t. the particle comes in only from the right,

$$\left.\begin{aligned} F &= \frac{i\Gamma}{1-i\Gamma}G \\ B &= \frac{1}{1-i\Gamma}G. \end{aligned}\right\} \text{—— (b.)}$$

When both $A \neq 0$ & $G \neq 0$, we can add contributions due to both (a.) & (b.),

s.t.,
$$B = \frac{i\Gamma}{1-i\Gamma}A + \frac{1}{1-i\Gamma}G.$$

$$F = \frac{1}{1-i\Gamma}A + \frac{i\Gamma}{1-i\Gamma}G.$$

$$\therefore \begin{pmatrix} B \\ F \end{pmatrix} = \frac{1}{(1-i\Gamma)}\begin{pmatrix} i\Gamma & 1 \\ 1 & i\Gamma \end{pmatrix}\begin{pmatrix} A \\ G \end{pmatrix}$$

$\therefore$ S-matrix for $V = -\alpha\,\delta(x)$ is

given by

$$S = \begin{pmatrix} \dfrac{i\Gamma}{(1-i\Gamma)} & \dfrac{1}{(1-i\Gamma)} \\[2ex] \dfrac{1}{(1-i\Gamma)} & \dfrac{i\Gamma}{(1-i\Gamma)} \end{pmatrix}$$

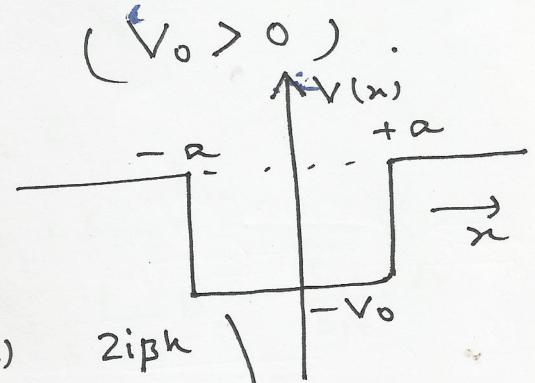
where, $\Gamma = \left(\dfrac{m\alpha}{\hbar^2 k}\right)$.

H.W. For Finite square well

$$V(x) = \begin{cases} -V_0, & \text{for } -a<x<a, \\ 0, & \text{for } |x|>a, \end{cases}$$

$(V_0 > 0)$.

show that:



$$S = \left(\frac{e^{-2ika}}{\sin(2\beta a)(k^2+\beta^2) + 2i\beta k\cos(2\beta a)}\right) \times \begin{pmatrix} (k^2-\beta^2)\sin(2\beta a) & 2i\beta k \\[2ex] 2i\beta k & (k^2-\beta^2)\sin(2\beta a) \end{pmatrix}.$$

where, $k = \sqrt{\dfrac{2mE}{\hbar^2}}$, $\beta = \sqrt{\dfrac{2m(E+V_0)}{\hbar^2}}$

M-matrix for $V(x) = -\alpha\delta(x)$.

We have,

$$\begin{pmatrix} B \\ F \end{pmatrix} = \underbrace{\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}}_{\text{S-matrix}} \begin{pmatrix} A \\ G \end{pmatrix}. \quad \text{— (i)}$$

By definition,

$$\begin{pmatrix} F \\ G \end{pmatrix} = \underbrace{\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}}_{\substack{\text{transfer matrix} \\ \text{or, M-matrix}}} \begin{pmatrix} A \\ B \end{pmatrix}. \quad \text{— (ii)}$$

From (i), $F = S_{21} A + S_{22} G$.

But $B = S_{11} A + S_{12} G$.

i.e., $G = \frac{1}{S_{12}} B - \frac{S_{11}}{S_{12}} A$.

$$\therefore \quad F = S_{21} A + \frac{S_{22}}{S_{12}} B - \frac{S_{11} S_{22}}{S_{12}} A$$

$$\text{i.e.,} \quad F = -\frac{(S_{11} S_{22} - S_{21} S_{12})}{S_{21}} A + \frac{S_{22}}{S_{21}} B. \quad \text{— (iii)}$$

Similarly, as $B = S_{11} A + S_{12} G$

($\&$ $F = S_{21} A + S_{22} G$.)

from def. of S-matrix

$$\cancel{\Rightarrow G = + \frac{1}{S_{12}}}$$

$$\Rightarrow G = -\frac{S_{11}}{S_{12}} A + \frac{1}{S_{12}} B.$$

$$\therefore \begin{pmatrix} F \\ G \end{pmatrix} = -\frac{1}{S_{12}} \begin{pmatrix} \det S & -S_{22} \\ S_{11} & -1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}.$$

$$\Rightarrow \boxed{M = -\frac{1}{S_{12}} \begin{pmatrix} \det S & -S_{22} \\ S_{11} & -1 \end{pmatrix}}$$

HW.

Evaluate M using

$$S = \begin{pmatrix} \frac{i\Gamma}{(1-i\Gamma)} & \frac{1}{(1-i\Gamma)} \\ \frac{1}{(1-i\Gamma)} & \frac{i\Gamma}{(1-i\Gamma)} \end{pmatrix}.$$

$$\Gamma = \frac{m\alpha}{\hbar^2 k}.$$

One can also show:

$$S = \frac{1}{M_{22}} \begin{pmatrix} -M_{21} & 1 \\ \det M & M_{12} \end{pmatrix}$$

HW.

Thus, for particle coming from left:

$$R_L = \frac{|B|^2}{|A|^2} \qquad (\because G = 0.)$$

$$= |S_{11}|^2 .$$

$$= \frac{|M_{21}|^2}{|M_{22}|^2} .$$

$$\& \quad T_L = \frac{|F|^2}{|A|^2} \qquad (\because G = 0.)$$

$$= |S_{21}|^2$$

$$= \frac{|\det M|^2}{|M_{22}|^2} .$$

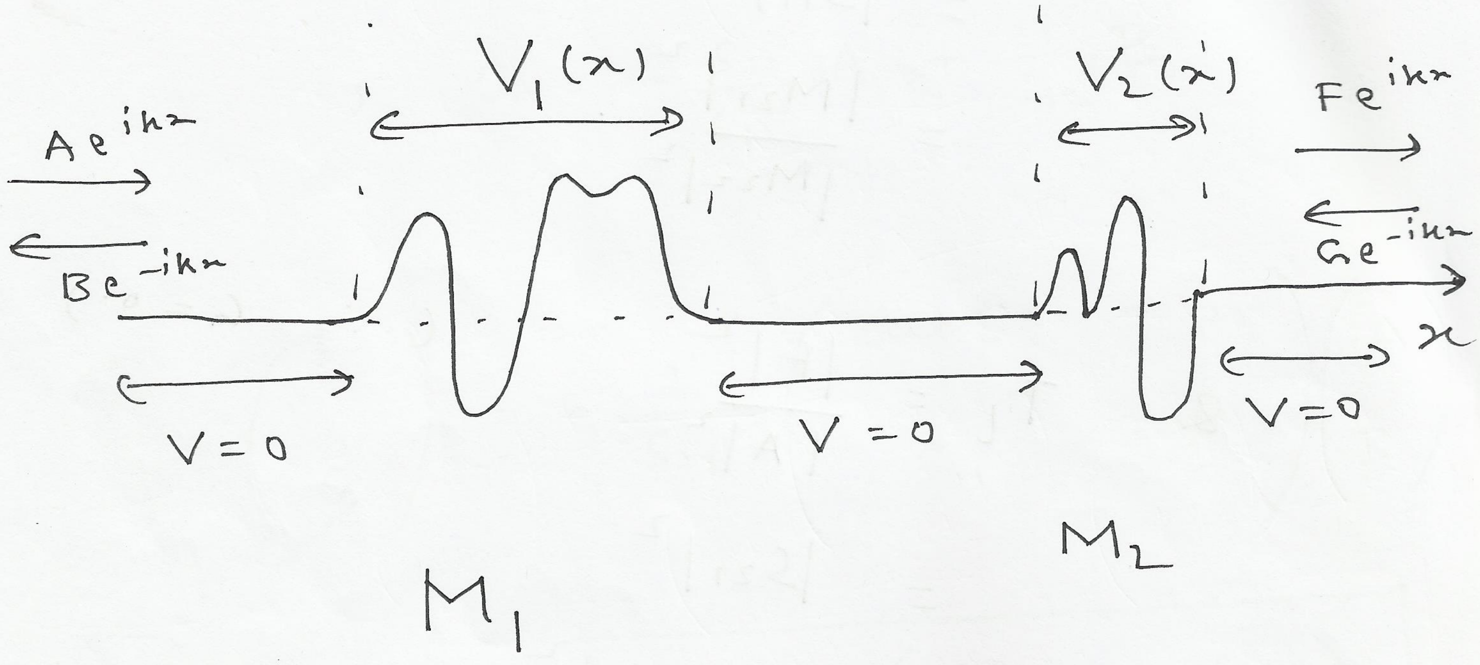
And for particle coming from right,

$$R_r = \frac{|F|^2}{|G|^2} = |S_{22}|^2 = \frac{|M_{12}|^2}{|M_{22}|^2}$$

$$T_r = \frac{|B|^2}{|G|^2} = |S_{12}|^2 = \frac{1}{|M_{22}|^2}$$

Significance/Utility of M-matrix

Suppose we have a potential consisting of two isolated pieces :



$$\begin{pmatrix} F \\ G \end{pmatrix} = M_2 M_1 \begin{pmatrix} A \\ B \end{pmatrix}.$$

i.e., $M = M_2 M_1$.

Using M one can obtain S & obtain R & T.

(X)

HW.

For $V(x) = -\alpha\left[\delta(x+a) + \delta(x-a)\right]$

show that:

$$M = \frac{1}{4k^2}\begin{bmatrix} z^2\left(e^{-4ika}-1\right)+4k^2+4ikz & i\left[4k\cos(2ka) - 2z\sin(2ka)\right] \\ -i\left(4k\cos(2ka) - 2z\sin(2ka)\right) & z^2\left(e^{4ika}-1\right)+4k^2-4ikz \end{bmatrix}$$

where,

$$M_1 = \frac{1}{2k}\begin{bmatrix} 2k+iz & ize^{-2ika} \\ ize^{2ika} & -2k+iz \end{bmatrix}.$$

$$M_2 = \frac{1}{2k}\begin{bmatrix} 2k+iz & ize^{2ika} \\ ize^{-2ika} & -2k+iz \end{bmatrix}.$$

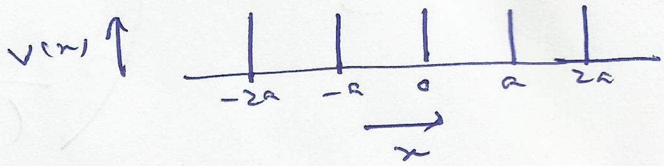
where, $z = \sqrt{\dfrac{2m\alpha}{\hbar^2}}$.

$k = \sqrt{\dfrac{2mE}{\hbar^2}}$.

Example (possible application of M-matrix & S-matrix formalism)

Dirac comb potential

$$V(x) = V_0 \sum_n \delta(x - na)$$



lattice spacing $\to 0$
$V_0 \to \infty$

$\Updownarrow$

Kronig-Peeney model

$\uparrow$ limiting case of periodic potential (as in a semiconductor)