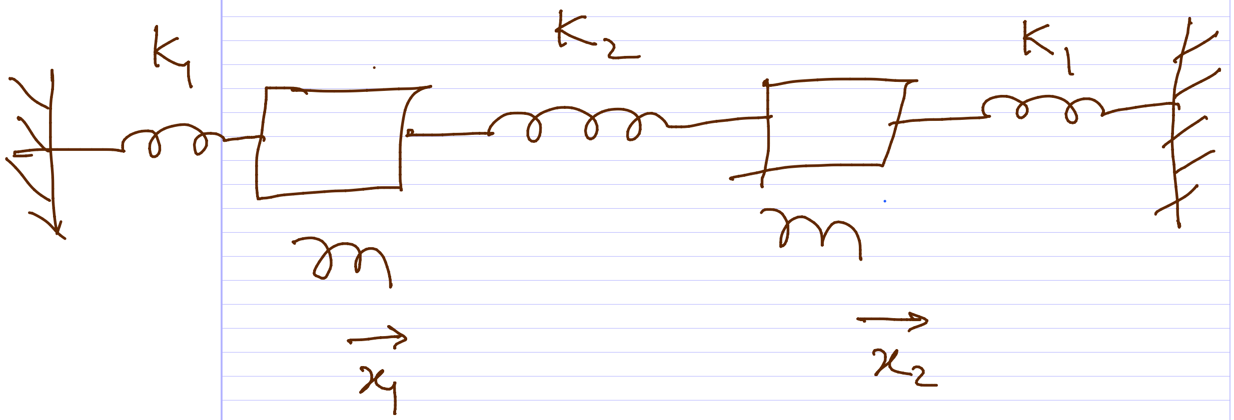


$$m\ddot{x}_1 = -K_1 x_1 - K_2(x_1 - x_2)$$

$$m\ddot{x}_2 = -K_1 x_2 - K_2(x_2 - x_1)$$

$$\Rightarrow \begin{cases} m\ddot{x}_1 = -(K_1+K_2)x_1 + K_2 x_2 \\ m\ddot{x}_2 = K_2 x_1 - (K_1+K_2)x_2 \end{cases}$$

$$\Rightarrow \underbrace{\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}}_{\overleftrightarrow{M}} \underbrace{\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix}}_{\ddot{\vec{X}}} = - \underbrace{\begin{pmatrix} K_1+K_2 & -K_2 \\ -K_2 & K_1+K_2 \end{pmatrix}}_{\overleftrightarrow{\mathbb{K}}} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{\vec{X}}$$

$$\overleftrightarrow{M}\,\ddot{\vec{X}} = -\overleftrightarrow{\mathbb{K}}\,\vec{X}.$$

$$\Rightarrow \ddot{\vec{X}} = -\overleftrightarrow{M}^{-1}\overleftrightarrow{\mathbb{K}}\,\vec{X}$$

But $\ddot{\vec{X}} = -\omega^2 \overleftrightarrow{\mathbb{1}}\,\vec{X}$.

$$\Rightarrow \overleftrightarrow{M}^{-1}\overleftrightarrow{K}\vec{x} = \omega^2 \overleftrightarrow{\mathbb{1}}\vec{x}.$$

$$\therefore \left(\overleftrightarrow{M}^{-1}\overleftrightarrow{K} - \omega^2\overleftrightarrow{\mathbb{1}}\right)\vec{x} = 0$$

$$\therefore \det\left|\overleftrightarrow{M}^{-1}\overleftrightarrow{K} - \omega^2\overleftrightarrow{\mathbb{1}}\right| = 0.$$

$\forall$ non-trivial $\vec{x}$.

$$\overleftrightarrow{M} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \Rightarrow \overleftrightarrow{M}^{-1} = \begin{pmatrix} \frac{1}{m} & 0 \\ 0 & \frac{1}{m} \end{pmatrix}$$

$$\overleftrightarrow{K} = \begin{pmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_1+k_2 \end{pmatrix}$$

$$\Rightarrow \overleftrightarrow{M}^{-1}\overleftrightarrow{K} = \begin{pmatrix} \frac{1}{m} & 0 \\ 0 & \frac{1}{m} \end{pmatrix}\begin{pmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_1+k_2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{k_1+k_2}{m} & -\frac{k_2}{m} \\ -\frac{k_2}{m} & \frac{k_1+k_2}{m} \end{pmatrix}.$$

$$\Rightarrow \begin{vmatrix} \frac{k_1+k_2}{m} - \omega^2 & -\frac{k_2}{m} \\ -\frac{k_2}{m} & \frac{k_1+k_2}{m} - \omega^2 \end{vmatrix} = 0.$$

$$\Rightarrow \left(\frac{k_1+k_2}{m} - \omega^2\right)^2 - \left(\frac{k_2}{m}\right)^2 = 0.$$

$$\Rightarrow (\omega^2)^2 - 2\left(\frac{k_1+k_2}{m}\right)\omega^2 + \frac{(k_1+k_2)^2 - k_2^2}{m^2} = 0.$$

$$\Rightarrow \quad \omega^2 = \frac{\frac{2(k_1+k_2)}{m} \pm \sqrt{\frac{4(k_1^2+k_2^2+2k_1k_2)}{m^2} - \frac{4k_1(k_1+2k_2)}{m^2}}}{2}$$

$$= \frac{k_1+k_2}{m} \pm \frac{1}{m}\sqrt{k_1^2+k_2^2+2k_1k_2 - k_1^2 - 2k_1k_2}$$

$$= \frac{k_1+k_2}{m} \pm \frac{k_2}{m}$$

$$= \begin{cases} \dfrac{k_1+2k_2}{m} \\ \dfrac{k_1}{m} \end{cases}$$

<u>Case I</u>: $\omega_1^2 = \dfrac{k_1+2k_2}{m}$.

Let $\vec{X} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} \frac{k_1+k_2}{m} & -\frac{k_2}{m} \\ -\frac{k_2}{m} & \frac{k_1+k_2}{m} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{k_1+2k_2}{m} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}.$$

$$\Rightarrow (k_1+k_2)a_1 - k_2 a_2 = (k_1+2k_2)a_1$$

$$(k_1+k_2)a_2 - k_2 a_1 = (k_1+2k_2)a_2$$

$$\Rightarrow a_2 = -a_1 = -a \quad (\text{say}).$$

$\Rightarrow$ Normalized

$$\vec{X}_1 = \frac{1}{\sqrt{2}\,a}\begin{pmatrix} a \\ -a \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Identify $\omega_1 = \omega_b = \sqrt{\dfrac{k_1 + 2k_2}{m}}$

$$\& \ |e_b\rangle = \vec{X}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

(breathing mode)

Case - II.

$$\omega_2^2 = \frac{k_1}{m} \ ; \ \text{Let } \vec{X}_2 = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

$$\Rightarrow \begin{pmatrix} \dfrac{k_1+k_2}{m} & -\dfrac{k_2}{m} \\ -\dfrac{k_2}{m} & \dfrac{k_1+k_2}{m} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \frac{k_1}{m} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

$$\Rightarrow (k_1 + k_2) b_1 - k_2 b_2 = k_1 b_1$$

$$-k_2 b_1 + (k_1 + k_2) b_2 = k_1 b_2$$

$$\Rightarrow b_1 = b_2 = b \quad (\text{say}).$$

$$\therefore \vec{X}_2 = \frac{1}{\sqrt{2}\, b} \begin{pmatrix} b \\ b \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Identify $\omega_2 = \omega_p = \sqrt{\dfrac{k_1}{m}}$. (pendulum mode)

$$\text{and } |e_p\rangle = \vec{X}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

$$\Rightarrow \langle e_b| = \frac{1}{\sqrt{2}} (1 \quad -1) \ , \ \langle e_p| = \frac{1}{\sqrt{2}} (1 \quad 1).$$

$$\therefore\ |e_p\rangle\langle e_p| + |e_b\rangle\langle e_b|$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\end{pmatrix}$$

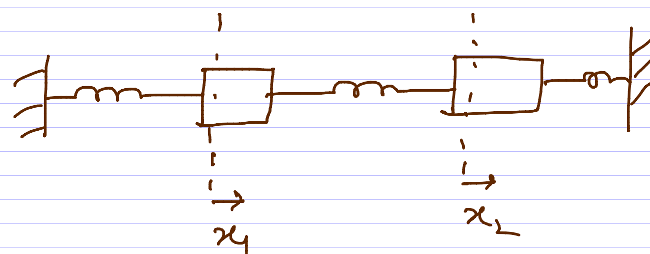
$$= \frac{1}{2}\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1 & -1\\-1 & 1\end{pmatrix}$$

$$= \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$$

$$= \underline{\overleftrightarrow{\Pi}}_2$$

(Illustrates the idea of complete set of states).

<u>Pendulum mode</u>



<u>Breathing mode</u>