Date

Matrix maths: (few preliminaries)

You will learn more in MAxxx.

- If $\det(\overleftrightarrow{A}) \neq 0$, then $\overleftrightarrow{A}\,\overleftrightarrow{A}^{-1} = \overleftrightarrow{A}^{-1}\overleftrightarrow{A} = \overleftrightarrow{\mathbb{1}}$.

  where, $\overleftrightarrow{\mathbb{1}} = \begin{pmatrix} 1 & 0 & 0 & 0 & \vdots \\ 0 & 1 & 0 & 0 & \vdots \\ 0 & 0 & 1 & 0 & \vdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$, identity matrix (all diagonal elements 1.) "rest zero"

- $\overleftrightarrow{\mathbb{1}}\,\overleftrightarrow{A} = \overleftrightarrow{A}$; $\overleftrightarrow{\mathbb{1}}$ operation leaves $\overleftrightarrow{A}$ unchanged

  (Thus, $\overleftrightarrow{A}^{-1}$ undoes the effect of $\overleftrightarrow{A}$.).

- Matrix multiplication is Associative

  i.e., $\overleftrightarrow{A}(\overleftrightarrow{B}\,\overleftrightarrow{C}) = (\overleftrightarrow{A}\,\overleftrightarrow{B})\overleftrightarrow{C}$.

  Example; $\langle g| = (g_1 \;\; g_2)$, $|x\rangle = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

  $\overleftrightarrow{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

  $\Rightarrow (\langle g|\overleftrightarrow{A})|x\rangle = \langle g|(\overleftrightarrow{A}|x\rangle)$.

- Hermitian matrices (useful in QM) : $\overleftrightarrow{A} = \overleftrightarrow{A}^{\dagger}$.

  (also called self adjoint matrices)

$[\overleftrightarrow{A}|x\rangle]^{\dagger} = ?$

Example: $\overleftrightarrow{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $|x\rangle = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$\Rightarrow \overleftrightarrow{A}^{\dagger} = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}$, $\langle x| = (|x\rangle)^{\dagger} = (x_1^* \;\; x_2^*)$.

$\Rightarrow \overleftrightarrow{A}|x\rangle = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{pmatrix}$

$\Rightarrow (\overleftrightarrow{A}|x\rangle)^{\dagger} = (a^*x_1^* + b^*x_2^* \quad c^*x_1^* + d^*x_2^*)$.

Oddy

$$\langle x|\overleftrightarrow{A}^{\dagger} = (x_1^* \;\; x_2^*)\begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}$$

$$= (a^* x_1^* + b^* x_2^* \quad c^* x_1^* + d^* x_2^*)$$

$$= (\overleftrightarrow{A}|x\rangle)^{\dagger}.$$

$\therefore\; (\overleftrightarrow{A}|x\rangle)^{\dagger} = \langle x|\overleftrightarrow{A}^{\dagger} = \langle x|\overleftrightarrow{A}$ (for Hermitian matrices).

<u>HW.</u> Which of these are Hermitian?

(i) $\begin{pmatrix} \omega_A^2 & -\frac{k}{m_1} \\ -\frac{k}{m_2} & \omega_B^2 \end{pmatrix}$, (ii) $\begin{pmatrix} \alpha i & 0 \\ 0 & \alpha \end{pmatrix}$

(iii) $\begin{pmatrix} -5 & \sqrt{2} \\ -\sqrt{2} & -1 \end{pmatrix}$

In fact a general 2×2 Hermitian matrix can be written as $\begin{pmatrix} \alpha & \beta \\ \beta^* & c \end{pmatrix}$

- For Hermitian matrices, $(\overleftrightarrow{A}\overleftrightarrow{B})^{\dagger} = \overleftrightarrow{B}^{\dagger}\overleftrightarrow{A}^{\dagger}$.

proof: Let $|\beta\rangle = \overleftrightarrow{B}|x\rangle$.

$$\Rightarrow (\overleftrightarrow{A}\overleftrightarrow{B}|x\rangle)^{\dagger} = (\overleftrightarrow{A}|\beta\rangle)^{\dagger} = \langle\beta|\overleftrightarrow{A}^{\dagger} = \langle\beta|\overleftrightarrow{A}$$

As $\overleftrightarrow{B}$ is Hermitian, $\langle\beta| \equiv (|\beta\rangle)^{\dagger} = (\overleftrightarrow{B}|x\rangle)^{\dagger}$
$= \langle x|\overleftrightarrow{B}^{\dagger} = \langle x|\overleftrightarrow{B}$.

$$\therefore (\overleftrightarrow{A}\overleftrightarrow{B}|x\rangle)^{\dagger} = \langle\beta|\overleftrightarrow{A} = \langle x|\overleftrightarrow{B}\overleftrightarrow{A}.$$

$$\Rightarrow (\overleftrightarrow{A}\overleftrightarrow{B})^{\dagger} = \overleftrightarrow{B}\overleftrightarrow{A}.$$

Date

Recast earlier results in vector space notation

$$\overleftrightarrow{M} \frac{d^2}{dt^2} |x(t)\rangle = -\overleftrightarrow{K} |x(t)\rangle .$$

If $|e_n\rangle$ is one of the eigenvectors.

$$\therefore \langle e_n | \overleftrightarrow{M} \frac{d^2}{dt^2} | x(t)\rangle = -\langle e_n | \overleftrightarrow{K} | x(t)\rangle .$$

$$\Rightarrow \frac{d^2}{dt^2}\left(\langle e_n | \overleftrightarrow{M} | x(t)\rangle\right) = -\langle e_n | \overleftrightarrow{K} | x(t)\rangle .$$

(Note! no time dep. of $\overleftrightarrow{M}$ & $\langle e_n|$.).

Define normal coordinates

$$S_n = \langle e_n | \overleftrightarrow{M} | x(t)\rangle .$$

$\therefore$ to claim a result like

$$\ddot{S}_n = -\omega_n^2 S_n ,$$

we must have, $\langle e_n | \overleftrightarrow{K} | x(t)\rangle = \omega_n^2 S_n$.

proof :- we had $\overleftrightarrow{M}^{-1} \overleftrightarrow{K} | e_n\rangle = \omega_n^2 | e_n\rangle$.

Adjoint : $\left[\overleftrightarrow{M}^{-1} \overleftrightarrow{K} | e_n\rangle\right]^\dagger = \omega_n^2 \langle e_n|$.

$$\Rightarrow \langle e_n | \overleftrightarrow{K} \overleftrightarrow{M}^{-1} = \omega_n^2 \langle e_n| .$$

Take inner product with $\overleftrightarrow{M} | x(t)\rangle$

$$\Rightarrow \langle e_n | \overleftrightarrow{K} \overleftrightarrow{M}^{-1} \overleftrightarrow{M} | x(t)\rangle = \omega_n^2 \langle e_n | \overleftrightarrow{M} | x(t)\rangle .$$

$\therefore \langle e_n | \mathbb{K} | x(t) \rangle = \omega_n^2 S_n .$

$\therefore$ $\boxed{S_n \equiv \langle e_n | \overleftrightarrow{M} | x(t) \rangle}$ is the normal coordinate.

(For equal masses).

Few results regarding the normalized eigenvectors

- $\langle e_m | e_n \rangle = 0$ if $n \neq m$

  (normal modes do not depend on each other; mutually orthogonal).

- $\langle e_n | e_n \rangle = 1$ (normalized).

In a Compact form it can be rewritten as,

$$\langle e_m | e_n \rangle = \delta_{mn}$$

where, Kronecker delta δ_{nm} is

$$\delta_{nm} = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}$$

Oddy