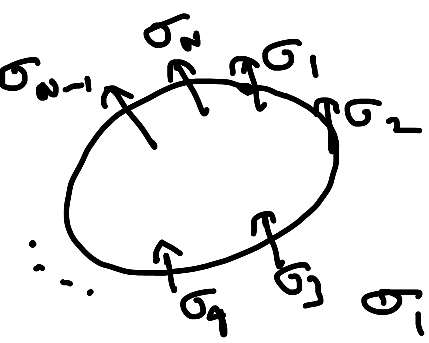
# Ising model

$$\mathcal{H} = -J\sum_{i=1}^{N}\sigma_i\sigma_{i+1} - h\sum_{i=1}^{N}\sigma_i$$

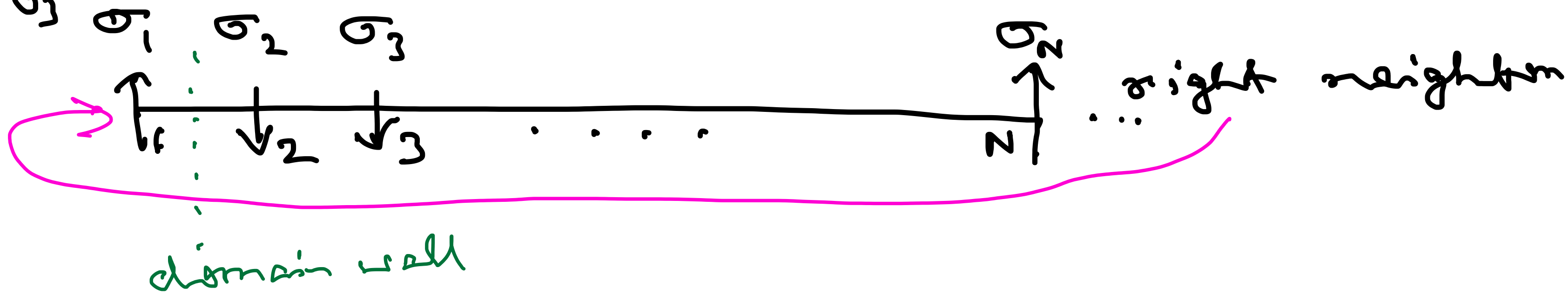
$$Z_N = \sum_{\{\sigma_i\}} \exp\left[K\sum_{i=1}^{N}\sigma_i\sigma_{i+1} + \frac{L}{2}\sum_{i=1}^{N}(\sigma_i+\sigma_{i+1})\right]$$

$$K = \beta J, \quad L = \beta h$$




Periodic Boundary conditions

$$\sigma_{N+1} = \sigma_1 .$$




$$Z = \sum_{\{\sigma_i\}}\prod_{i=1}^{N} T_{\sigma_i,\sigma_{i+1}}$$ ← Transfer matrix/Density matrix

where, $T_{\sigma_i,\sigma_{i+1}} = \exp\left[K\sigma_i\sigma_{i+1} + \frac{L}{2}(\sigma_i+\sigma_{i+1})\right]$

$$T = \begin{array}{c c} & \sigma' \rightarrow \quad +1 \qquad\quad -1 \\ \begin{array}{c}\sigma\downarrow \\ +1 \\ \\ -1\end{array} & \begin{pmatrix} e^{K+L} & e^{-K} \\ e^{-K} & e^{K-L}\end{pmatrix}\end{array} = \begin{pmatrix} e^{K+L} & e^{-K} \\ e^{-K} & e^{K-L}\end{pmatrix}$$

$$Z = \mathrm{Tr}\, D^N$$ ← diagonalized

where, $D$ is the diagonalized version of $T$.

$$U T U^{-1} = D \quad (\text{unitary transf}^n.)$$

$$U^{-1} = U^{\dagger}.$$

$$\therefore \quad Z = \mathrm{Tr}(\underbrace{U^{-1} D U}_{T})^N$$

$$= \mathrm{Tr}(U^{-1} D \underbrace{U U^{-1}}_{\mathbb{I}} D U \ldots\ldots U^{-1} D U) \quad N \text{ times}$$

$$= \mathrm{Tr}(U^{-1} D^N U)$$

$$= \mathrm{Tr}(D^N U U^{-1}) \qquad (\because \mathrm{Tr}(AB) = \mathrm{Tr}(BA).)$$

$$= \mathrm{Tr}\, D^N.$$

The problem of finding $Z$ reduces to diagonalization of $T$.

# Diagonalization of T

$$T = \begin{pmatrix} e^{K+L} & e^{-K} \\ e^{-K} & e^{(K-L)} \end{pmatrix}$$

$\lambda$'s s.t., $\det(T - \lambda \mathbb{I}) = 0$

$$\Rightarrow \left(e^{K+L} - \lambda\right)\left(e^{K-L} - \lambda\right) - e^{-2K} = 0$$

$$\therefore \lambda^2 - \left(e^{K+L} + e^{K-L}\right)\lambda + \left(e^{2K} - e^{-2K}\right) = 0.$$

$$\lambda_{\{1,2\}} = \frac{\left(e^{K+L} + e^{K-L}\right) \pm \sqrt{e^{2(K+L)} + e^{2(K-L)} + 2e^{2K} - 4\left(e^{2K} - e^{-2K}\right)}}{2}$$

$$= e^{K} \cosh L \pm \sqrt{\left[e^{2K}\cosh^2[L] - 2\sinh(2K)\right]}.$$

[Check: Both $\lambda_1$ & $\lambda_2 > 0$].
(Exercise)

$$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad \& \quad D^N = \begin{pmatrix} \lambda_1^N & 0 \\ 0 & \lambda_2^N \end{pmatrix}$$

$$Z = Tr(D^N)$$

Trace $\equiv$ sum of diagonal elements of $D^N$.

$$= (\lambda_1^N + \lambda_2^N)$$

Rest of the thermodynamic quantities can be obtained from $Z$.

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## Case : $h = 0$

$$\Rightarrow L = 0 \quad , \quad L = \beta h$$

$$\cosh L = \frac{e^L + e^{-L}}{2} = \frac{(1 + L + \frac{L^2}{2!} \cdots) + (1 - L + \frac{L^2}{2!} - + \cdots)}{2}$$

$$= 1 + L^2 + \ldots$$

$$\longrightarrow 1 \quad \text{as } L \to 0 .$$

$$\therefore \lambda_{\{\frac{1}{2}\}} = e^k \pm \sqrt{e^{2k} - 2\sinh(2K)}$$

$$= e^k \pm \sqrt{e^{2k} - 2\frac{(e^{2k} - e^{-2k})}{2}}$$

$$= e^k \pm e^{-k}$$

$$= \begin{cases} 2\cosh k \\ 2\sinh k \end{cases}$$

$\therefore \lambda_1 = 2\cosh K \quad , \lambda_2 = 2\sinh K.$

$\lambda_1 \geq \lambda_2$

$$Z = \lambda_1^N + \lambda_2^N = \lambda_1^N\left[1 + \left(\frac{\lambda_2}{\lambda_1}\right)^N\right].$$

$$f = \lim_{N\to\infty} \frac{F}{N} = \lim_{N\to\infty}\left[-\frac{1}{\beta N}\ln Z\right] \approx -\frac{1}{\beta}\ln\lambda_1.$$

$(\lambda_2 < \lambda_1)$.

$$\therefore f \approx -\frac{1}{\beta}\ln(2\cosh K). \quad (\text{for } h=0)$$

<u>General case ($h \neq 0$):</u>

$$f = -\frac{1}{\beta}\ln\underbrace{\left\{e^{\beta J}\cosh(\beta h) + \sqrt{e^{2\beta J}\cosh^2(\beta h) - 2\sinh(2\beta J)}\right\}}_{\lambda_1}$$

$$\left(\because \left(\frac{\lambda_2}{\lambda_1}\right)^N \to 0 \text{ as } N\to\infty.\right)$$

$$m = -\left(\frac{\partial f}{\partial h}\right)_T = \frac{\sinh(\beta h)}{\sqrt{[\sinh^2(\beta h) + \exp(-4\beta J)]}}$$

$$\xrightarrow[h\to 0]{} 0$$

(1-d Ising model cannot explain ferromagnetism).

Find $s = -\left(\frac{\partial f}{\partial T}\right)_h$

$c_h = \left(\frac{\partial U}{\partial T}\right)_h = T\left(\frac{\partial s}{\partial T}\right)_h$

H.W.

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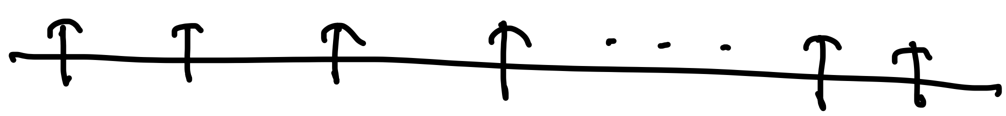
Imagine a situation



Try create a domain wall

Energy cost $\Delta U = 2J > 0$.

Subsequently, change in entropy

$$\Delta S = k_B \ln(N-1).$$

$$\Delta F = \underbrace{2J - k_B \ln(N-1)}_{}.$$

$< 0$ for sufficiently large N,

as free energy decreases, more domain walls are formed,

$\Rightarrow$ disordered phase.

Spin-spin correlation $\langle \sigma_j . \sigma_k \rangle_{N\to\infty}$

$$\langle \sigma_j \sigma_k \rangle_N = \frac{1}{Z} \sum_{\{\sigma\}} \sigma_j \sigma_k e^{-\beta \mathcal{H}}$$

Homework

$$\xrightarrow[k>j]{} \frac{\lambda_1^{N-(k-j)} \lambda_2^{(k-j)} + \lambda_1^{(k-j)} \lambda_2^{N-(k-j)}}{\lambda_1^N + \lambda_2^N} .$$

$$\xrightarrow[N\to\infty]{} \left(\frac{\lambda_2}{\lambda_1}\right)^{k-j}$$

$$\langle \sigma_j \sigma_k \rangle \xrightarrow[\substack{N\to\infty \\ h\to 0}]{} (\tanh K)^R \quad ; \quad R = |K-j| .$$

$$= \exp(R \ln(\tanh K)) = e^{-R/\xi}$$

distance.

where, $\xi = \dfrac{1}{|\ln(\tanh K)|}$,

correlation length.

Absence of order in 2d

Bethe-ansatz for Heisenberg

Further reading.

$\xi$ diverges for $K\to\infty$, i.e., $T\to 0$.

$T=0$, trivial fixed point
(No ordered phase at $T\neq 0$.).

For $T\neq 0$, the spin-spin correlation decays exponentially.