Random walk

Specific model for a walker in 1-d
- Step length l
- step interval τ
- probability of moving to right p
- probability of moving to left q
- $p+q=1$

Let there be N_1 steps to right & N_2 steps to the left, s.t.,

$$N = N_1 + N_2 \equiv \text{total number of steps.}$$

and, $m = N_1 - N_2 \equiv$ displacement to the right (in units of l).

Therefore, the probability that the walker has a displacement ml after N steps,
$$P_N(m) = \underbrace{\frac{N!}{N_1!\,N_2!}p^{N_1}q^{N_2}}_{W_N(N_1)} = \frac{N!}{\left(\frac{N+m}{2}\right)!\left(\frac{N-m}{2}\right)!}p^{\left(\frac{N+m}{2}\right)}q^{\left(\frac{N-m}{2}\right)}$$

$$\langle N_1\rangle = \sum_{N_1=0}^{N} N_1 \frac{N!}{N_1!\,N_2!}p^{N_1}q^{N_2} = p\frac{\partial}{\partial p}\left\{\sum_{N_1=0}^{N}\frac{N!}{N_1!\,N_2!}p^{N_1}q^{N_2}\right\}$$

But, $\displaystyle\sum_{N_1=0}^{N}\frac{N!}{N_1!\,N_2!}p^{N_1}q^{N_2} = (p+q)^N.$

$$\Rightarrow \langle N_1\rangle = Np(p+q)^{N-1} = Np \quad (\because p+q=1.)$$

Similarly, $\langle N_1^2\rangle = \left(p\frac{\partial}{\partial p}\right)^2(p+q)^N$

$$= Np(p+q)^{N-1} + N(N-1)p^2(p+q)^{N-2}$$

$$= Np + N(N-1)p^2$$

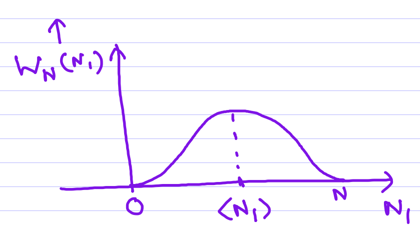
$$\Rightarrow \langle(\Delta N_1)^2\rangle = \langle N_1^2\rangle - \langle N\rangle^2 = Np + N(N-1)p^2 - N^2p^2$$

$$= Np(1-p) = Npq.$$

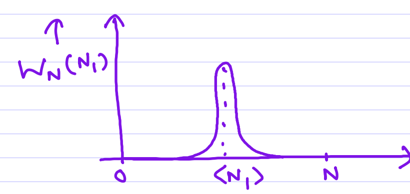
$$\Rightarrow \sqrt{\langle(\Delta N_1)^2\rangle} = \sigma_{N_1} = \sqrt{Npq}.$$

$$\therefore \frac{\sigma_{N_1}}{\langle N_1\rangle} = \frac{\sqrt{Npq}}{Np} = \frac{1}{\sqrt{N}}\sqrt{\left(\frac{q}{p}\right)}.$$

Thus, for $q=p=\frac{1}{2}$, $\dfrac{\sigma_{N_1}}{\langle N_1\rangle} = \dfrac{1}{\sqrt{N}}$.



$\sigma_{N_1} \sim \langle N_1\rangle$



$\sigma_{N_1} \ll \langle N_1\rangle$

Generating function for Binomial distribution

$$f(x) = \sum_{N_1=0}^{N} W_N(N_1)\, x^{N_1} = (px+q)^N .$$

Partially differentiating w.r.t. x on both sides, and setting $x=1$,

$$\Rightarrow \sum_{N_1=0}^{N} N_1 x^{N_1-1} W_N(N_1)\Big|_{x=1} = Np(px+q)^{N-1}\Big|_{x=1} .$$

$$\Rightarrow \langle N_1 \rangle = Np .$$

Similarly, taking a second partial derivative w.r.t. x and setting $x=1$,

$$\Rightarrow \langle N_1(N_1-1) \rangle = N(N-1)p^2 .$$

$$\therefore \langle N_1^2 \rangle - \langle N_1 \rangle = N(N-1)p^2$$

$$\Rightarrow \langle N_1^2 \rangle = N^2p^2 - Np^2 + Np \qquad (\because \langle N_1 \rangle = Np.)$$

$$\therefore \sigma_{N_1}^2 = (\Delta N_1)^2 = \langle N_1^2 \rangle - \langle N_1 \rangle^2 = N^2p^2 - Np^2 + Np - N^2p^2 .$$

$$= Np(1-p) = Npq .$$

$$\Rightarrow \sigma_{N_1} = \sqrt{Npq} .$$

Binomial distribution $\longrightarrow$ Gaussian distribution

Recap: Stirling approximation $\quad n! \sim n^n e^{-n}\sqrt{2\pi n}$.
large n

$$n! = \int_0^\infty x^n e^{-x}\,dx = \int_0^\infty e^{(n\ln x - x)}\,dx$$

Let $f(x) = n\ln x - x$.

$$f'(x) = \frac{n}{x} - 1.$$

$$f''(x) = -\frac{n}{x^2}.$$

$$f(n) = n\ln n - n$$

$$f'(n) = 0$$

$$f''(n) = -\frac{1}{n}$$

Expanding $f(x)$ in a Taylor series about n,

$$\Rightarrow f(x) \approx n\ln n - n - \frac{1}{2n}(x-n)^2.$$

$$\therefore\ n! \sim \int_0^\infty e^{\ln n^n} e^{-n} e^{-(x-n)^2/2n}\,dx = n^n e^{-n}\int_0^\infty e^{-\frac{(x-n)^2}{2n}}\,dx$$

Let $x - n = y \Rightarrow dx = dy$

For $x = 0$, $y = -n$

For $x = \infty$, $y = \infty$

$\therefore$ For large enough n

$$n! \sim n^n e^{-n}\int_{-n}^\infty e^{-y^2/2n}\,dy$$

$$\sim \sqrt{2\pi n}\, n^n e^{-n}.$$

$$\therefore\ \ln n! \approx n\ln n - n.$$

$$W_N(N_1) = \frac{N!}{N_1!\,(N-N_1)!}\, p^{N_1} q^{N-N_1}$$

$$\Rightarrow \ln W_N(N_1) = \ln N! - \ln N_1! - \ln(N-N_1)! + N_1\ln p + (N-N_1)\ln q.$$

$$= N\ln N - N_1\ln N_1 - (N-N_1)\ln(N-N_1) + N_1\ln p + (N-N_1)\ln q.$$

$$\therefore\ \left.\frac{\partial \ln W_N(N_1)}{\partial N_1}\right|_{N_1=\tilde{N}_1} = -1 - \ln\tilde{N}_1 + 1 + \ln(N-\tilde{N}_1) + \ln p - \ln q.$$

$$= 0 \Rightarrow \frac{N-\tilde{N}_1}{\tilde{N}_1} = \frac{q}{p} \Rightarrow \tilde{N}_1 = Np.$$

i.e., most probable value equals average value.

$$(\tilde{N}_1 = \langle N_1\rangle = Np)$$

Similarly, $\dfrac{\partial^2 \ln W_N(N_1)}{\partial N_1^2} = -\dfrac{1}{N_1} - \dfrac{1}{N-N_1}$

$$\therefore \left.\frac{\partial^2 \ln W_N(N_1)}{\partial N_1^2}\right|_{N_1=\tilde{N}_1=Np} = -\frac{1}{Np} - \frac{1}{Nq} = -\frac{1}{Npq}.$$

$\therefore$ Using Taylor series expansion,

$$\ln W_N(N_1) \approx \ln W_N(\tilde{N}_1) - \frac{1}{2Npq}(N_1-\tilde{N}_1)^2$$

$$\therefore W_N(N_1) \approx W_N(\tilde{N}_1) \exp\left\{-\frac{(N_1-\langle N_1\rangle)^2}{2\sigma_{N_1}^2}\right\}$$

(Gaussian distribution.).

Alternate derivation:

$$W_N(N_1) = \frac{N!}{N_1!\,(N-N_1)!}\, p^{N_1} q^{N-N_1}.$$

Use Stirling approximation, $n! \sim n^n e^{-n}\sqrt{2\pi n}\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right)$.

$$\Rightarrow W_N(N_1) \approx \frac{N^N e^{-N}\sqrt{2\pi N}\; p^{N_1} q^{N-N_1}}{N_1^{N_1} e^{-N_1}\sqrt{2\pi N_1}\,(N-N_1)^{N-N_1} e^{-(N-N_1)}\sqrt{2\pi(N-N_1)}}\left(1+\mathcal{O}\left(\frac{1}{N}\right)\right).$$

$$= \frac{N^N}{N_1^{N_1}(N-N_1)^{N-N_1}}\sqrt{\frac{N}{2\pi N_1(N-N_1)}}\; p^{N_1} q^{N-N_1}\left(1+\mathcal{O}\left(\frac{1}{N}\right)\right).$$

$$= \left(\frac{Np}{N_1}\right)^{N_1}\left(\frac{Nq}{N-N_1}\right)^{N-N_1}\sqrt{\frac{N}{2\pi N_1(N-N_1)}}\left(1+\mathcal{O}\left(\frac{1}{N}\right)\right). \quad \text{— (I)}$$

Let, $\delta = N_1 - Np \quad \Rightarrow N - N_1 = N(1-p) - \delta = Nq - \delta.$

$$\therefore \ln\left(\frac{Np}{N_1}\right) = \ln\left(\frac{Np}{Np+\delta}\right) = -\ln\left(1+\frac{\delta}{Np}\right)$$

$$\approx -\left(\frac{\delta}{Np} - \frac{\delta^2}{2N^2p^2} + \mathcal{O}\left(\frac{\delta^3}{N^3p^3}\right)\right).$$

$$\& \quad \ln\left(\frac{Nq}{N-N_1}\right) = \ln\left(\frac{Nq}{Nq-\delta}\right) = -\ln\left(1-\frac{\delta}{Nq}\right)$$

$$\approx \left(\frac{\delta}{Nq} + \frac{\delta^2}{2N^2q^2} + \mathcal{O}\left(\frac{\delta^3}{N^3q^3}\right)\right).$$

$$\Rightarrow \ln\left[\left(\frac{Np}{N_1}\right)^{N_1}\left(\frac{Nq}{N-N_1}\right)^{N-N_1}\right] = -(\delta+Np)\left(\frac{\delta}{Np} - \frac{\delta^2}{2N^2p^2} + \mathcal{O}\left(\frac{\delta^3}{N^3p^3}\right)\right)$$

$$+ (Nq-\delta)\left(\frac{\delta}{Nq} + \frac{\delta^2}{2N^2q^2} + \mathcal{O}\left(\frac{\delta^3}{N^3q^3}\right)\right)$$

$$= \delta(-1+1) + \delta^2\left[\left(\frac{1}{2Np}+\frac{1}{2Nq}\right) - \frac{1}{Np} - \frac{1}{Nq}\right] + \mathcal{O}(\delta^3).$$

$$= -\frac{\delta^2}{2Npq} + \mathcal{O}(\delta^3). \quad \text{— (II)}$$

Also, $$\sqrt{\frac{N}{2\pi N_1(N-N_1)}} = \sqrt{\frac{N}{2\pi(\delta+Np)(Nq-\delta)}} = \sqrt{\frac{1}{2\pi Npq - 2\pi(p-q)\delta - \frac{2\pi\delta^2}{N}}}$$

$$\approx \sqrt{\frac{1}{2\pi Npq}}. \quad \text{— (III)}$$

Using (I), (II) & (III), $$\Rightarrow W_N(N_1) \approx \sqrt{\frac{1}{2\pi Npq}}\exp\left(-\frac{\delta^2}{2Npq}\right) = \sqrt{\frac{1}{2\pi Npq}}\exp\left[-\frac{(N_1-Np)^2}{2Npq}\right].$$

Random walk & Diffusion equation.

$$P_{N+1}(m) = p\, P_N(m-1) + q\, P_N(m+1). \quad \text{---} \; (a)$$

$\therefore$ For $p = q = \frac{1}{2}$, $\Rightarrow P_{N+1}(m) = \frac{1}{2} P_N(m-1) + \frac{1}{2} P_N(m+1)$.

Now,

$$\lim_{\tau \to 0} \frac{P_{N+1}(m) - P_N(m)}{\tau} = \frac{\partial P}{\partial t}. \quad \text{---} \; (b)$$

we can also construct the difference table,

$P_N(m-1)$		
	$P_N(m) - P_N(m-1)$	
$P_N(m)$		$P_N(m+1) + P_N(m-1) - 2P_N(m)$
	$P_N(m+1) - P_N(m)$	
$P_N(m+1)$		

$$\Rightarrow \lim_{l \to 0} \frac{P_N(m+1) + P_N(m-1) - 2P_N(m)}{l^2} = \frac{\partial^2 P}{\partial x^2} \quad \text{---} \; (c)$$

From (a),

$$P_{N+1}(m) - P_N(m) = \frac{1}{2} P_N(m+1) + \frac{1}{2} P_N(m-1) - P_N(m)$$

$$\Rightarrow \frac{P_{N+1}(m) - P_N(m)}{\tau} = \frac{l^2}{2\tau} \, \frac{P_N(m+1) + P_N(m-1) - 2P_N(m)}{l^2}. \quad \text{---} \; (d)$$

$\therefore$ using (b) & (c) in (d),

$$\Rightarrow \frac{\partial P}{\partial t} = D \frac{\partial^2 P}{\partial x^2} \qquad \text{with, } D = \frac{l^2}{2\tau}.$$

Poisson distribution as a special case of Binomial distribution.

$$W_N(N_1) = \frac{N!}{N_1!\,(N-N_1)!}\, p^{N_1}(1-p)^{N-N_1}.$$

- Large number of trials
- Probability of success in a trial is small.

Define, $\lambda = Np$.

$$\therefore \lim_{\substack{N\to\infty \\ p\to 0}} W_N(N_1) = \frac{N!}{N_1!\,(N-N_1)!}\left(\frac{\lambda}{N}\right)^{N_1}\left(1-\frac{\lambda}{N}\right)^{N-N_1}.$$

Note:

- $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e.$
- Also, $N_1 = k \ll N.$

$$P(\lambda,k) = \lim_{\substack{N\to\infty \\ p\to 0 \\ N_1 = k}} W_N(N_1).$$

$$= \lim_{\frac{\lambda}{N}\to 0} \frac{N!}{k!\,(N-k)!}\left(\frac{\lambda}{N}\right)^k\left(1-\frac{\lambda}{N}\right)^{N-k}.$$

$$= \lim_{\frac{\lambda}{N}\to 0} \underbrace{\frac{N(N-1)\cdots(N-k+1)}{N^k}}_{\to 1}\left(\frac{\lambda^k}{k!}\right)\left(1-\frac{\lambda}{N}\right)^N \underbrace{\left(1-\frac{\lambda}{N}\right)^{-k}}_{\to 1}.$$

(finite terms $N\to\infty$)

$$= \frac{\lambda^k}{k!} \lim_{\frac{N}{\lambda}\to\infty}\left(1+\frac{1}{\left(-\frac{N}{\lambda}\right)}\right)^{-\lambda\left(-\frac{N}{\lambda}\right)}.$$

$$= \frac{\lambda^k}{k!}\, e^{-\lambda}.$$

$$\therefore \boxed{P(\lambda,k) = \frac{\lambda^k}{k!}\, e^{-\lambda}}$$

k trials $\ll N$.

$\lambda = Np$, $p \to$ small.