Quiz - I (February 19, 2024)

Hints & solution

① Ergodic hypothesis:

- Non-zero recurrence probability
- Time average = Ensemble average

The answer must include these aspects

② $p(x) = \frac{1}{6\alpha}\exp(-|x|/3\alpha)$

$$\Rightarrow \tilde{p}(k) = \frac{1}{6\alpha}\int_{-\infty}^{\infty} dx\, \exp\left(-ikx - \frac{|x|}{3\alpha}\right)$$

$$= \frac{1}{6\alpha}\left[\int_{0}^{\infty} dx\, \exp\left(-ikx - \frac{x}{3\alpha}\right) + \int_{-\infty}^{0} dx\, \exp\left(-ikx + \frac{x}{3\alpha}\right)\right]$$

$$= \frac{1}{6\alpha}\left[-\frac{1}{\left(-ik - \frac{1}{3\alpha}\right)} + \frac{1}{\left(-ik + \frac{1}{3\alpha}\right)}\right] = \frac{1}{6\alpha}\frac{3\alpha(2)}{(1+3ik\alpha)(1-3ik\alpha)}$$

$$= \frac{1}{1+9k^2\alpha^2} = 1 - 9k^2\alpha^2 + 81k^4\alpha^4 - + \cdots$$

But $\tilde{p}(k) = \sum_{n=0}^{\infty}\frac{(-ik)^n}{n!}\langle x^n\rangle = 1 - ik\langle x\rangle - \frac{k^2}{2!}\langle x^2\rangle - \cdots$

By comparison, $\langle x\rangle = 0$.

$\langle x^2\rangle = 18\alpha^2$.

③ $E = -\frac{\mu_0 H}{2}N_1 + \frac{\mu_0 H}{2}(N - N_1)$

$$\Rightarrow N_1 = \frac{1}{2}\left(N - \frac{2E}{\mu_0 H}\right) \quad \& \quad N_2 = N - N_1 = \frac{1}{2}\left(N + \frac{2E}{\mu_0 H}\right).$$

ⓐ $$\Omega(E,N) = \frac{N!}{N_1!\,N_2!} = \frac{N!}{\left[\frac{1}{2}\left(N - \frac{2E}{\mu_0 H}\right)\right]!\,\left[\frac{1}{2}\left(N + \frac{2E}{\mu_0 H}\right)\right]!}$$

ⓑ Let $u = \frac{E}{N}$ be internal energy per particle.

$$\Rightarrow \underset{\substack{E,N\to\infty \\ (E/N)=u}}{\mathrm{Lt}} \frac{1}{N}\ln\Omega(E,N) = \frac{1}{N}\Big[N\ln N - \cancel{N} - \left(\frac{N}{2} - \frac{Nu}{\mu_0 H}\right)\ln\left(\frac{N}{2} - \frac{Nu}{\mu_0 H}\right) - \left(\frac{N}{2} + \frac{Nu}{\mu_0 H}\right)\ln\left(\frac{N}{2} + \frac{Nu}{\mu_0 H}\right) + \cancel{\frac{N}{2}} + \cancel{\frac{N}{2}}\Big] + O(\ln N).$$

$$= \ln N - \frac{1}{2}\left(1 - \frac{2u}{\mu_0 H}\right)\ln\left[\frac{N}{2}\left(1 - \frac{2u}{\mu_0 H}\right)\right] - \frac{1}{2}\left(1 + \frac{2u}{\mu_0 H}\right)\ln\left[\frac{N}{2}\left(1 - \frac{2u}{\mu_0 H}\right)\right]$$