Ref:- J.A. Pelesko & D.H. Bernstein
Modeling MEMS & NEMS
(Chapman & Hall / CRC)

ADT

Pressure sensor ⟹ deflects

Tweezers ⟹ bend

Optical switches ⟹ twist

Pumps ⟹ flexes

Accelerometers ⟹ ...

& so on

Elasticity (Mechanical) in MEMS/NEMS

Stress & Strain

Strain → ϵ_{ij}

Stress → σ_{ij} (3 × 3 matrix)

- The ij^{th} component, σ_{ij} denotes the force per unit area in the direction i exerted on a surface element with normal in the j direction.

$$\sigma_{ij} = \sigma_{ji} \quad \text{(symmetric)}$$

⟸ no net angular momentum.

∴ 6 independent components.

- For small displacements,

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

u_i is the i^{th} component of the displacement vector.

- In general,

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \underbrace{\frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}}_{\text{quadratic term neglected for small displ.}}\right).$$

Apply conservation of momentum (to appropriate control volume)*:

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial \sigma_{ij}}{\partial x_j} + f_i . \quad \text{— I}$$

body force per unit volume

(3 eqn.)

* Analogous to the application of conservation of energy while deriving heat transfer eqn.

Using generalized Hooke's laws,

$$\sigma_{ij} = 2\mu \epsilon_{ij} + \lambda \epsilon_{kk} \delta_{ij}.$$

μ, λ are Lame constants

s.t.,

$$\underset{\text{(Youngs mod.)}}{Y} = \frac{\mu(2\mu + 3\lambda)}{\lambda + \mu}$$

$$\& \quad \underset{\text{(Poisson ratio)}}{\nu} = \frac{\lambda}{2(\lambda + \mu)}$$

[without proof.].

Thus,

$$Y \epsilon_{ij} = (1+\nu)\sigma_{ij} - \nu \sigma_{kk} \delta_{ij}. \quad \text{(II)}$$

use (II) in (I),

$$\Rightarrow \rho \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial}{\partial x_j}(2\mu \epsilon_{ij} + \lambda \epsilon_{kk}\delta_{ij}).$$

Eliminating ϵ_{ij},

Navier eqn. of Linear elasticity

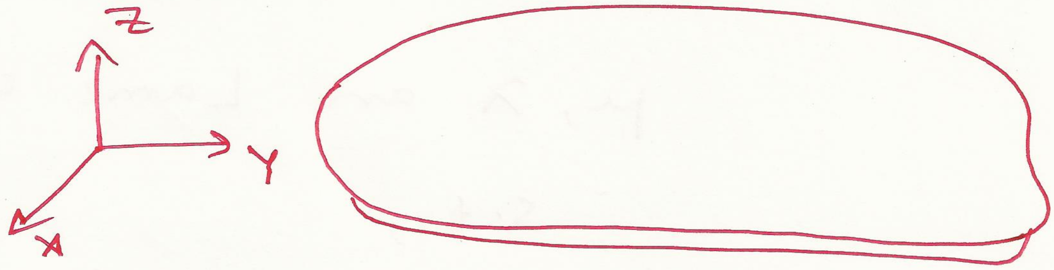
Gov. Eqn.

$$\Rightarrow \rho \frac{\partial^2 \vec{u}}{\partial t^2} = \mu \nabla^2 \vec{u} + (\mu + \lambda)\nabla(\nabla \cdot \vec{u}) + \vec{f}.$$

(+ I.C. + B.C.) — (III)

Example :

(A). Membrane (thin)



Assume that membrane is only allowed to deflect in the x_3 (or, z) direction.

Displacement vector $\begin{pmatrix} 0 \\ 0 \\ u_3 \end{pmatrix}$.

For thin membrane

$$u_3 \equiv u_3(x, y, t).$$

$\therefore$ Navier's eq.

$$\Rightarrow \rho \frac{\partial^2 u_3}{\partial t^2} = \mu \nabla^2 u_3 .$$

$$\text{s.t.,} \quad \nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} .$$

Initial conditions:

Displacement $u_3(x, y, 0) = f(x, y)$

velocity $\frac{\partial u_3}{\partial t}(x, y, 0) = g(x, y)$

Boundary condition!

$$u_3(x, y, t) = 0 \quad \text{on boundary}$$

(Dirichlet).

More interesting versions

Elastic waves $\xrightarrow{\text{dissipates}}$ heat

Thermal gradient $\xrightarrow{\text{induces}}$ stress

$\therefore$ Eqn of linear thermoelasticity

Governing Eqn.

$$\rho C_p \frac{\partial T}{\partial t} + \nabla \cdot \mathbb{K} \nabla T - \alpha(3\lambda + 2\mu) T_0 \frac{\partial \epsilon_{kk}}{\partial t}.$$

α = coeff. of thermal expansion

T_0 = ref. temperature.

$\Updownarrow$

Duhamel-Neumann (modified Hooke's law)

$$\sigma_{ij} = 2\mu \epsilon_{ij} + \lambda \delta_{ij} \epsilon_{kk} - \delta_{ij} \alpha (3\lambda + 2\mu)(T - T_0).$$

- rise in temp. creates stress