

# GOVERNING EQUATIONS - III

ADT

(1)

Passive MEMS structures: Inkjet nozzles, etc

Active MEMS structures: Micropumps, etc



Microfluidics



Non-linear PDEs

(Navier-Stokes eq.).

GOVERNING EQUATIONS

Conservation of mass

$$\Rightarrow \frac{\partial \rho}{\partial t} + u_j \frac{\partial \rho}{\partial x_j} + \rho \frac{\partial u_i}{\partial x_i} = 0.$$

↓ velocities ←

Define strain-rate tensor,

$$\frac{\partial \epsilon_{ij}}{\partial t} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right). \quad \text{--- (I)}$$

(Note:  $u_i \rightarrow$  velocities).

Stress in the fluid is described by  $\sigma_{ij}$ ,

s.t.,

$$\sigma_{ij} = -p \delta_{ij} + 2\mu \dot{\epsilon}_{ij} + \lambda \dot{\epsilon}_{kk} \delta_{ij}. \quad \text{--- (II)}$$

$p \Rightarrow$  pressure in the fluid.

$\mu \Rightarrow$  dynamic fluid viscosity.

$\lambda \Rightarrow$  second viscosity coefficient.

Conservation of momentum  $\Rightarrow$

$$\rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j} = \rho F_i + \frac{\partial \sigma_{ij}}{\partial x_j} \quad \text{--- (IV)}$$

mass  $\times$  acceleration
forces

for an arbitrary volume.

Using (III) in (IV),

$$\Rightarrow \rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j} = \rho F_i - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j} (2\mu \dot{\epsilon}_{ij} + \lambda \dot{\epsilon}_{kk} \delta_{ij}),$$

--- (V)

Eqn. (I) & (V) are called the Navier-Stokes equations,

In vector form, they are,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0 \quad \text{--- (VI)}$$

$$\rho \frac{\partial \vec{u}}{\partial t} + \rho (\vec{u} \cdot \nabla) \vec{u} + \nabla p - \mu \nabla^2 \vec{u} - (\lambda + \mu) \nabla (\nabla \cdot \vec{u}) = \vec{f}$$

System of four coupled non-linear <sup>partial</sup> differential equations for a viscous compressible fluid, --- (VII)

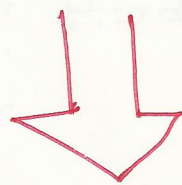


## Special case:

Incompressible NS equation for a flow with characteristic velocity  $U$ , in a spatial region of characteristic length  $l$ .

$$\vec{u}^* = \frac{\vec{u}}{U}, \quad x^* = \frac{x}{l}, \quad y^* = \frac{y}{l}, \quad z^* = \frac{z}{l},$$

$$t^* = \frac{Ut}{l}, \quad p^* = \frac{p - p_\infty}{\rho U^2}.$$



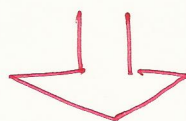
dimensionless system

$$\nabla \cdot \vec{u}^* = 0$$

$$\frac{\partial \vec{u}^*}{\partial t^*} + (\vec{u}^* \cdot \nabla) \vec{u}^* = -\nabla p^* + \frac{1}{\text{Re}} \nabla^2 \vec{u}^*.$$

$$\text{Re} = \frac{Ul}{\nu}.$$

$\text{Re} = 0 \Rightarrow$  Stokes equations.



Of particular interest to Microfluidics.

## Boundary conditions

At ~~solid~~ solid-fluid boundary:

No-penetration B.c.

$$\vec{u} \cdot \hat{n} = 0.$$

No-slip B.c.

$$\vec{u} \times \hat{n} = 0.$$

Together  $\Rightarrow \vec{u} = 0$  at boundary.

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## Electrodynamics. (brief recap).

$$\nabla \times \vec{E} =$$

$$\nabla \times \vec{B} =$$

$$\nabla \cdot \vec{E} =$$

$$\nabla \cdot \vec{B} =$$

maxwell's equations.

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}).$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0.$$

$$D_i = \epsilon_{ij} E_j.$$

$$B_i = \mu_{ij} H_j.$$

B.c.

$$(\vec{D}_2 - \vec{D}_1) \cdot \hat{n} = \sigma.$$

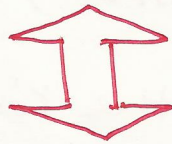
$$(\vec{B}_2 - \vec{B}_1) \cdot \hat{n} = 0.$$

$$(\vec{E}_2 - \vec{E}_1) \times \hat{n} = 0.$$

$$(\vec{H}_2 - \vec{H}_1) \times \hat{n} = -\vec{J}_s.$$

Note :

1. Five unknown functions viz.,  $p$ ,  $\rho$  and three components of  $\vec{u}$  (velocity vector field),
2. Conservation of energy  $\Rightarrow$  1 more equation & 1 unknown function  $T$  (temperature),
3. Introduce equation of state which relates  $\rho$ ,  $p$  &  $T$ .



Define the problem completely.

Analytic solution:

Elusive

(otelbaev)

Mukhtarbay Otelbayev of Eurasian

National University in Astana, Kazakhstan

proposed a solution in 2013.

(One flaw detected & community is after it.)



CASE I: Incompressible fluid.

$\rho$  is a constant.

$\therefore$  Navier-Stokes equations yield,

(VIII.) 
$$\left\{ \begin{array}{l} \nabla \cdot \bar{u} = 0 \\ \frac{\partial \bar{u}}{\partial t} + (\bar{u} \cdot \nabla) \bar{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \bar{u}. \end{array} \right.$$

where,  $\nu = \frac{\mu}{\rho}$  is

the kinematic viscosity.

(system of four equations in four unknowns).

CASE II: Inviscid & incompressible fluid

$\nu = 0$ ,  $\rho = \text{constant}$ .

$\therefore$  Navier-Stokes equations reduce to:

(Euler equation)

(IX.) 
$$\left\{ \begin{array}{l} \nabla \cdot \bar{u} = 0 \\ \frac{\partial \bar{u}}{\partial t} + (\bar{u} \cdot \nabla) \bar{u} = -\frac{1}{\rho} \nabla p. \end{array} \right.$$