Coordinate system $\rightleftharpoons$ frames of reference

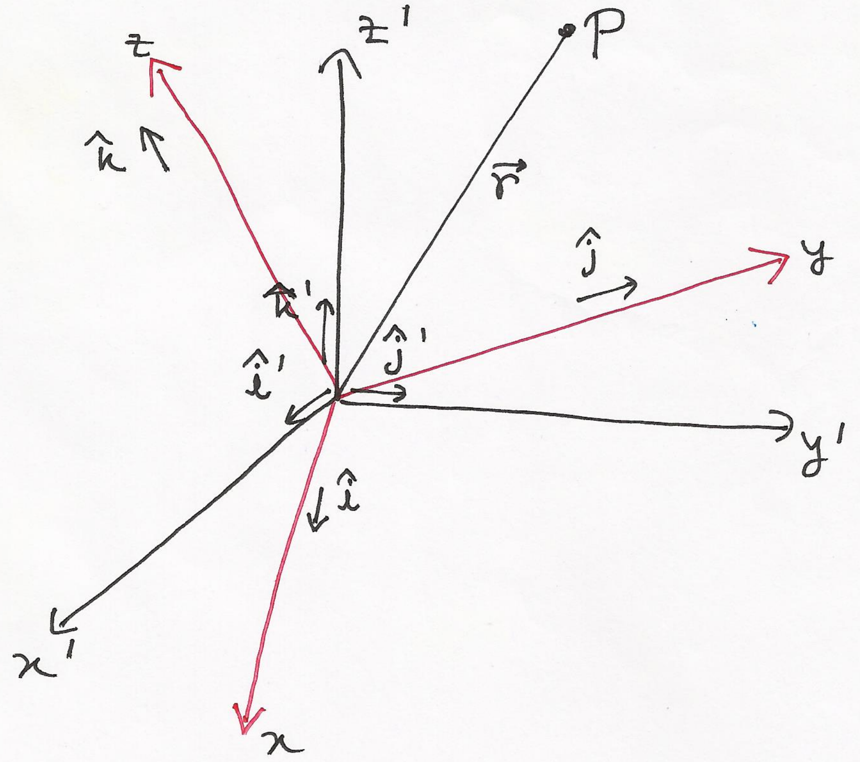
Frame of reference includes all the coordinate systems at rest w.r.t. any particular system.

If we use a moving frame of reference, a term occurs in the equation of motion due to the acceleration of the frame. To have the 'equation of motion' invariant under change of frame, we need to add a term $-m\ddot{\vec{R}}$ to the r.h.s.

This additional term represents a 'fictitious' / 'pseudo' / non-inertial force & has its existence due to the moving frame of reference.

e.g., centrifugal force felt by a person sitting inside a car when the car takes a sudden left turn.

Consider an unprimed coordinate system - O(x, y, z) which is rotating with angular velocity $\vec{\omega}$ about some instantaneous axis passing through the origin O. Here, O is the common origin of the rotating frame of reference — O(x, y, z) & the fixed frame of reference O'(x', y', z').



Position vector of a particle P can be written as,

$$\vec{r} = \hat{i}'x' + \hat{j}'y' + \hat{k}'z'.$$

$$\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z.$$

Transformation equations from unprimed to primed system can be obtained by taking the dot product of $\vec{r}$ with $\hat{i}'$, $\hat{j}'$ and $\hat{k}'$.

These are,

$$x' = (\vec{r}\cdot\hat{i}') = (\hat{i}\cdot\hat{i}')x + (\hat{j}\cdot\hat{i}')y + (\hat{k}\cdot\hat{i}')z$$

$$y' = (\vec{r}\cdot\hat{j}') =$$

$$z' = (\vec{r}\cdot\hat{k}') =$$

the dot products are equal to the cosines of angles between the axes.

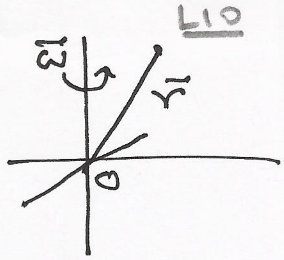
This is also true in general for any vector $\vec{V}(t)$.

$$V = \hat{i}V_x + \hat{j}V_y + \hat{k}V_z = \hat{i}'V_x' + \hat{j}'V_y' + \hat{k}'V_z'.$$

Time derivative would however be different in the two systems.

$$\left(\frac{dV}{dt}\right)_{fix} = \hat{i}'\dot{V}_x' + \hat{j}'\dot{V}_y' + \hat{k}'\dot{V}_z'.$$

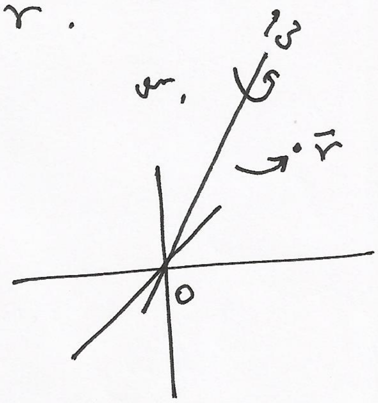
$$= \overbrace{\hat{i}\dot{V}_x + \hat{j}\dot{V}_y + \hat{k}\dot{V}_z}^{\text{time derivative in rotating system } \left(\frac{dV}{dt}\right)_{rot}} + \frac{d\hat{i}}{dt}V_x + \frac{d\hat{j}}{dt}V_y + \frac{d\hat{k}}{dt}V_z. \quad \text{(I)}$$



In particular,

$$\vec{v} = \frac{d\vec{r}}{dt} = \vec{\omega} \times \vec{r}.$$

or,



$\therefore$ For unit vectors $\hat{i}, \hat{j}, \hat{k}$ in rotating frame,

$$\frac{d\hat{i}}{dt} = \vec{\omega} \times \hat{i}, \quad \frac{d\hat{j}}{dt} = \vec{\omega} \times \hat{j}, \quad \frac{d\hat{k}}{dt} = \vec{\omega} \times \hat{k}. \qquad \text{(II.)}$$

Using (I.) & (II.),

$$\Rightarrow \left(\frac{d\vec{V}}{dt}\right)_{fix} = \left(\frac{d\vec{V}}{dt}\right)_{rot} + \vec{\omega} \times \vec{V}.$$

$$\boxed{\left(\frac{d}{dt}\right)_{fix} = \left(\frac{d}{dt}\right)_{rot} + \vec{\omega} \times} \qquad \text{(III.)}$$

Can be operated on any vector.

e.g.,
$$\left(\frac{d\vec{\omega}}{dt}\right)_{fix} = \left(\frac{d\vec{\omega}}{dt}\right)_{rot} + \vec{\omega} \times \vec{\omega}.$$
$$= \left(\frac{d\vec{\omega}}{dt}\right)_{rot}$$
$$= \dot{\vec{\omega}}.$$

$\Big\{$ same in fixed & rotating frames. — (IV.)

What about second derivatives? (5) 40

$$\left(\frac{d}{dt}\right)_{fix} = \frac{d'}{dt}$$

Notation

$$\left(\frac{d}{dt}\right)_{rot} = \frac{d}{dt}$$

$$\Rightarrow \frac{d'^2\vec{V}}{dt^2} = \frac{d'}{dt}\left(\frac{d'\vec{V}}{dt}\right)$$

$$= \frac{d'}{dt}\left[\frac{d\vec{V}}{dt} + \vec{\omega}\times\vec{V}\right]$$

$$= \left(\frac{d}{dt} + \vec{\omega}\times\right)\left(\frac{d\vec{V}}{dt} + \vec{\omega}\times\vec{V}\right).$$

$$= \cancel{\frac{d^2\vec{V}}{dt^2} + \vec{\omega}\times\frac{d\vec{V}}{dt}}$$

$$= \frac{d^2}{dt^2}\vec{V} + \frac{d\vec{\omega}}{dt}\times\vec{V} + \vec{\omega}\times\frac{d\vec{V}}{dt}$$

$$+ \vec{\omega}\times\frac{d\vec{V}}{dt} + \vec{\omega}\times\vec{\omega}\times\vec{V}. \quad \text{(V.)}$$

Consider the general case when the origin O of the rotating coordinate system is also moving w.r.t. O' of fixed frame.

$$\vec{r}\,' = \vec{R} + \vec{r}.$$

$$\therefore \left(\frac{d\vec{r}\,'}{dt}\right)_{fix} = \left(\frac{d\vec{R}}{dt}\right)_{fix} + \left(\frac{d\vec{r}}{dt}\right)_{fix}. \quad \text{(VI.)}$$

Using (III.), (IV.) & (V.) in (VI.),

show.

$$\left(\frac{d\vec{r}\,'}{dt}\right)_{fix} = \left(\frac{d\vec{R}}{dt}\right)_{fix} + \left(\frac{d\vec{r}}{dt}\right)_{rot} + \vec{\omega}\times\vec{r}.$$

&

$$\left(\frac{d^2\vec{r}\,'}{dt^2}\right)_{fix} = \left(\frac{d^2\vec{R}}{dt^2}\right)_{fix} + \left(\frac{d^2\vec{r}}{dt^2}\right)_{rot} + 2\vec{\omega}\times\left(\frac{d\vec{r}}{dt}\right)_{rot} + \vec{\omega}\times\vec{\omega}\times\vec{r} + \frac{d\vec{\omega}}{dt}\times\vec{r}.$$

Using symbols f & r for fixed & rotating frames,

$$\boxed{\begin{aligned} \dot{\vec{r}}\,'_f &= \dot{\vec{R}}_f + \dot{\vec{r}}_r + \vec{\omega}\times\vec{r} \\ \ddot{\vec{r}}\,'_f &= \ddot{\vec{R}}_f + \ddot{\vec{r}}_r + 2\vec{\omega}\times\dot{\vec{r}}_r + \vec{\omega}\times\vec{\omega}\times\vec{r} + \dot{\vec{\omega}}\times\vec{r}. \end{aligned}}$$

— VII A & B.

$\dot{\vec{r}}_f{}'$ & $\ddot{\vec{r}}_f{}'$ $\equiv$ velocity & accl^n. relative to fixed axes.

$\dot{\vec{R}}_f$ & $\ddot{\vec{R}}_f$ $\equiv$ linear velocity & linear accl^n. of the origin of the rotating axes.

$\dot{\vec{r}}_r$ & $\ddot{\vec{r}}_r$ = velocity & accl^n. relative to the rotating axes.

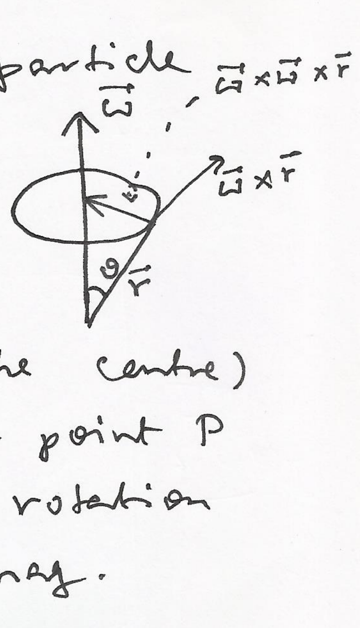
$\vec{\omega}$ = angular velocity of the rotating axes.

$\vec{\omega} \times \vec{r}$ = velocity due to rotation of the axes

$2\vec{\omega} \times \dot{\vec{r}}_r$ = Coriolis acceleration.

$\vec{\omega} \times \vec{\omega} \times \vec{r}$ = centripetal acceleration

$\dot{\vec{\omega}} \times \vec{r}$ = angular accl^n. of the particle due to accl^n. of the rotating axes.



<u>Note:</u> centripetal (meaning towards the centre) accl^n. of the particle situated at point P is directed towards the axes of rotation & is perpendicular to it with mag.

$|\vec{\omega} \times \vec{\omega} \times \vec{r}| = \omega^2 r \sin\theta = \dfrac{v^2}{r\sin\theta}$.

Note: Coriolis accl^{n} is present only when the particle has a velocity $\dot{\vec{r}}_r$ in the rotating frame.

Newton's second law,

$\vec{F} = m\ddot{\vec{r}}$ is only valid in the inertial frame of reference, $\vec{F} = \left(\frac{d^2\vec{r}}{dt^2}\right)_{fix}$.

What abt. eq. of motion of a particle in the rotating frame?

Assume: ∘ Angular velocity of rotating system is constant. $\dot{\vec{\omega}} = 0$.

∘ Origins of two frames coincide. $\vec{R} = 0$.

Using (VII.) B.,

$$m\left(\frac{d^2\vec{r}}{dt^2}\right)_{rot} = m\left(\frac{d^2\vec{r}}{dt^2}\right)_{fix} - 2m\vec{\omega}\times\left(\frac{d\vec{r}}{dt}\right)_{rot} - m\vec{\omega}\times(\vec{\omega}\times\vec{r}) = \vec{F}_{eff}.$$

Forces acting on the particle

in the rotating frame are:

(i.) Real force $m\left(\frac{d^2\vec{r}}{dt^2}\right)_{fix}$

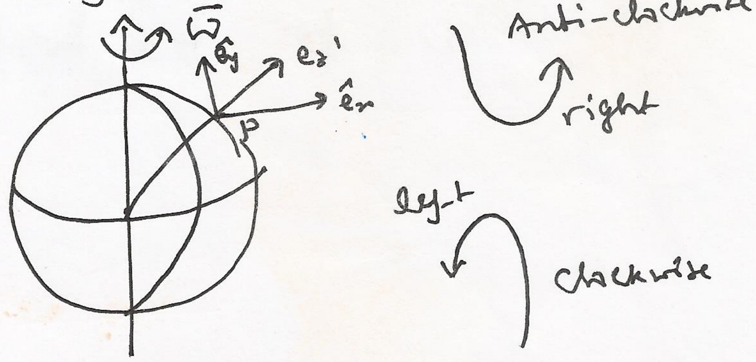
(ii.) Centrifugal force $-m\vec{\omega}\times(\vec{\omega}\times\vec{r})$

result of rotation of coordinate axes.

(iii.) Coriolis force $-2m\vec{\omega}\times\left(\frac{d\vec{r}}{dt}\right)_{rot}$.

result of motion of the particle in the rotating frame of reference.

→ Non-inertial forces.



Coriolis force

- Climate

For a velocity

$1\,km/s = 3600\,km/hr$

coriolis acc^n. $\approx 0.15\,m/sec$

$\cong 0.015\,g$.

- Flight of missiles

Ball on a turntable

Ball on a turntable

Example : Ship

Translation

Moving up & down (heaving)

" left & right (swaying)

" forward & backward (surging)

Rotation

Tilt forward & backward (pitching)

Swivels left & right (yawing)

Pivots side to side (rolling)

Robotics

Serial & parallel manipulator system : Designed to position an end effector with 6 DOF (3 for translation & 3 for rotation).

Actuator positions

$\Updownarrow$

Manipulator configuration.

> 6 DOF ; Dean Kamen (2007)

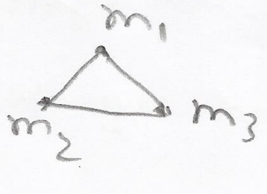
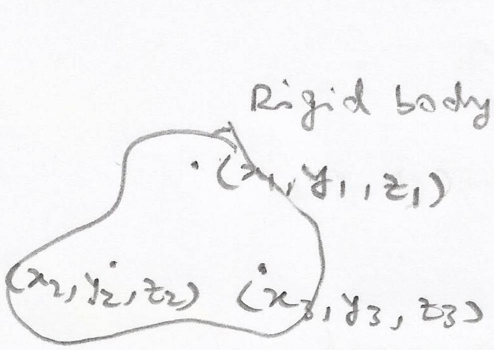
$\downarrow$

Robotic arm with 14 DOF.

DOF

(degrees of freedom)

$\equiv$ # of coordinates − # of constraint eq.

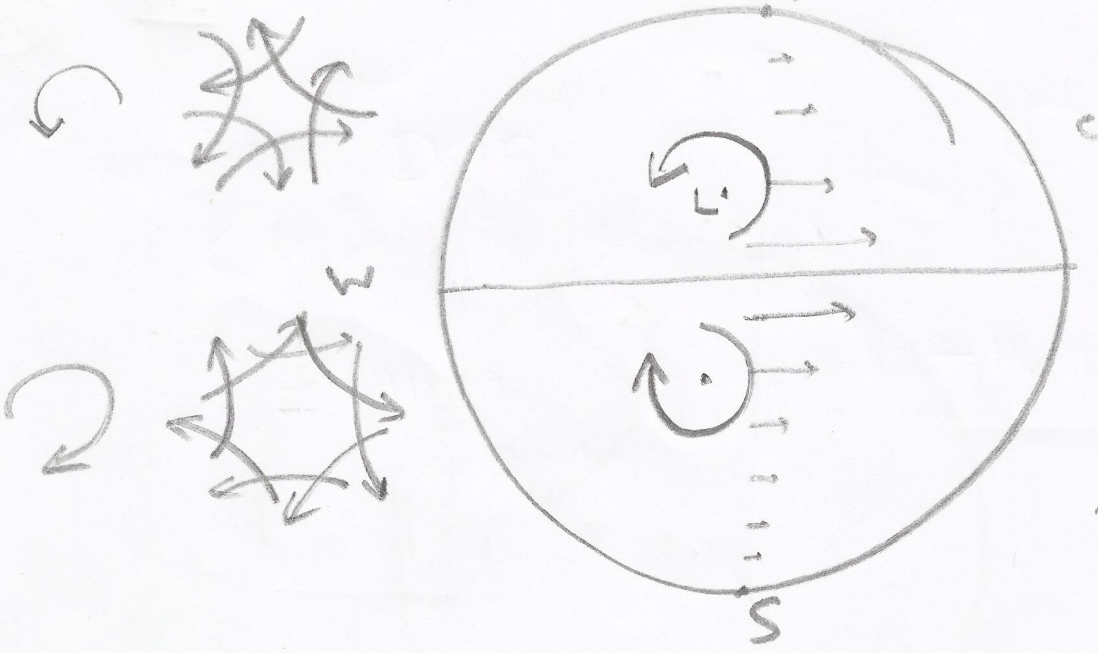
System	DOF
$\cdot\, m_1$	3
	$6-1=6$
	$9-3=6$
	$9-3=6$

const. eq.

$$\left\{ \begin{array}{l} (x_i-x_j)^2+(y_i-y_j)^2+(z_i-z_j)^2 = d_{ij}^2 . \\ \hookrightarrow 3 \text{ eq.} \end{array} \right.$$

Curve to the right N Hurricane

cyclone
Typhoon.



Tornadoe

Wind pattern