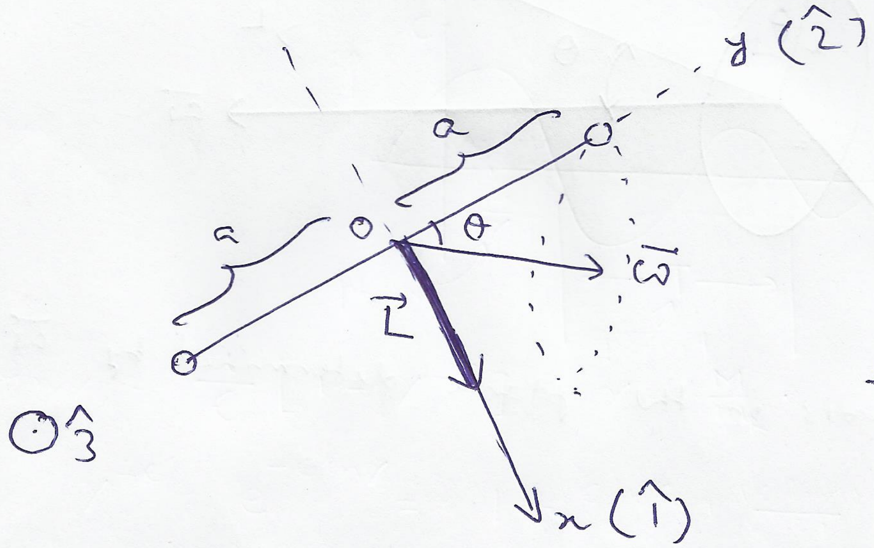
Two particle rotator



x axis lies in the plane containing y-axis & $\vec{\omega}$.

$$I_x = 2ma^2, \quad I_y = 0, \quad I_z = 2ma^2.$$

$$\omega_x = \omega \sin\theta, \quad \omega_y = \omega\cos\theta, \quad \omega_z = 0.$$

$$\therefore \quad \vec{L} = 2ma^2\omega \sin\theta \, \hat{x}.$$

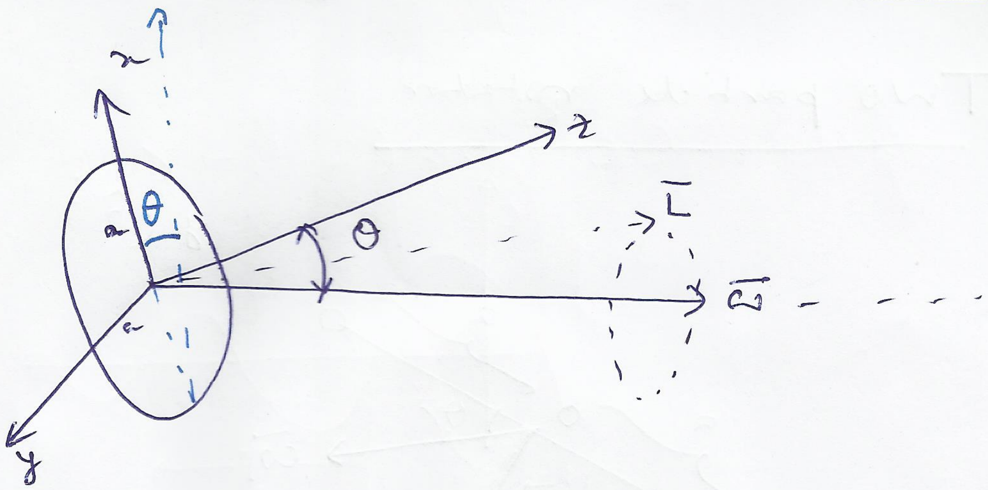
Using Euler's eq.,

$$I_z \frac{d\omega_z}{dt} + (I_y - I_x)\,\omega_x\omega_y = \tau_z.$$

$$\Rightarrow \tau_z = -2ma^2\omega^2 \sin\theta\cos\theta.$$

L12 - L13

Circular dish : Fixed axis inclined from normal



x-axis in the plane determined by $\bar{\omega}$ & $\hat{z}$.

$$I_x = I_y = \frac{Ma^2}{4} \; ; \; I_z = \frac{Ma^2}{2}.$$

$$\omega_x = -\omega \sin\theta, \quad \omega_y = 0, \quad \omega_z = \omega\cos\theta.$$

$$\Rightarrow \bar{L} = -\frac{1}{4} Ma^2 \omega \sin\theta \, \hat{x} + \frac{1}{2} Ma^2 \omega \cos\theta \, \hat{z}.$$

Angle α between $\bar{\omega}$ & $\bar{L}$ is,

$$\alpha = \cos^{-1}\left(\frac{\bar{\omega} \cdot \bar{L}}{|\bar{\omega}||\bar{L}|} \right) = \frac{1 + \cos^2\theta}{1 + 3\cos^2\theta}.$$

As motion progresses, $\bar{L}$ rotates about $\bar{\omega}$ (generating a cone).

From Euler eq., torque reqd. to hold the disk in rotation,

$$\bar{\tau} = \tau_y \hat{y} = \frac{1}{4} Ma^2 \omega^2 \sin\theta \cos\theta \, \hat{y}.$$

$$\alpha = \cos^{-1}\left(\frac{\vec{\omega}\cdot\vec{L}}{|\vec{\omega}||\vec{L}|}\right).$$

$$\frac{\vec{\omega}\cdot\vec{L}}{|\vec{\omega}||\vec{L}|} = \frac{\frac{1}{4}Ma^2\omega^2(1+\cos^2\theta)}{\omega\,\frac{1}{4}Ma^2\omega(\sin^2\theta+4\cos^2\theta)}.$$

$$= \frac{1+\cos^2\theta}{1+3\cos^2\theta}.$$

$$\therefore \quad \alpha = \cos^{-1}\left(\frac{1+\cos^2\theta}{1+3\cos^2\theta}\right).$$

Belongs to systems that are "not dynamically balanced".

$\vec{L}$ does not coincide in direction with $\vec{\omega}$ $\Rightarrow$ a (rotating) torque is reqd. to hold the body in rotation.

The dynamic balancing of crankshafts, wheels, etc., is required so that the axes of rotation coincide with the principal axes.

How is wheel balancing done?

By adjusting mass distributions.

Stability of rotational motion

Example: Book spinning about its three axes.

Let the body is initially spinning with $\omega_1 = \text{const}$ & $\omega_2 = \omega_3 = 0$.

After a brief perturbation, $\omega_2, \omega_3 \neq 0$.

When perturbation ends, $\omega_2, \omega_3 << \omega_1$ (small perturb.). motion is torque free.

$$\Rightarrow \quad I_1 \frac{d\omega_1}{dt} + (I_3 - I_2)\,\omega_2 \omega_3 = 0 \quad \text{—(i)}$$

$$I_2 \frac{d\omega_2}{dt} + (I_1 - I_3)\,\omega_3 \omega_1 = 0 \quad \text{—(ii)}$$

$$I_3 \frac{d\omega_3}{dt} + (I_2 - I_1)\,\omega_1 \omega_2 = 0 \quad \text{—(iii)}$$

$$\omega_2, \omega_3 << 1 \quad \Rightarrow \quad I_1 \frac{d\omega_1}{dt} = 0 \Rightarrow \omega_1 = \text{const}$$

$\therefore$ (ii) & (iii) $\Rightarrow$

$$I_2 \frac{d^2\omega_2}{dt^2} - \frac{(I_1 - I_3)(I_2 - I_1)}{I_3}\,\omega_1^2 \omega_2 = 0.$$

or,

$$\frac{d^2\omega_2}{dt^2} + A\omega_2 = 0.$$

where,

$$A = \frac{(I_1 - I_2)(I_1 - I_3)}{I_2 I_3}\,\omega_1^2$$

+ DEMO

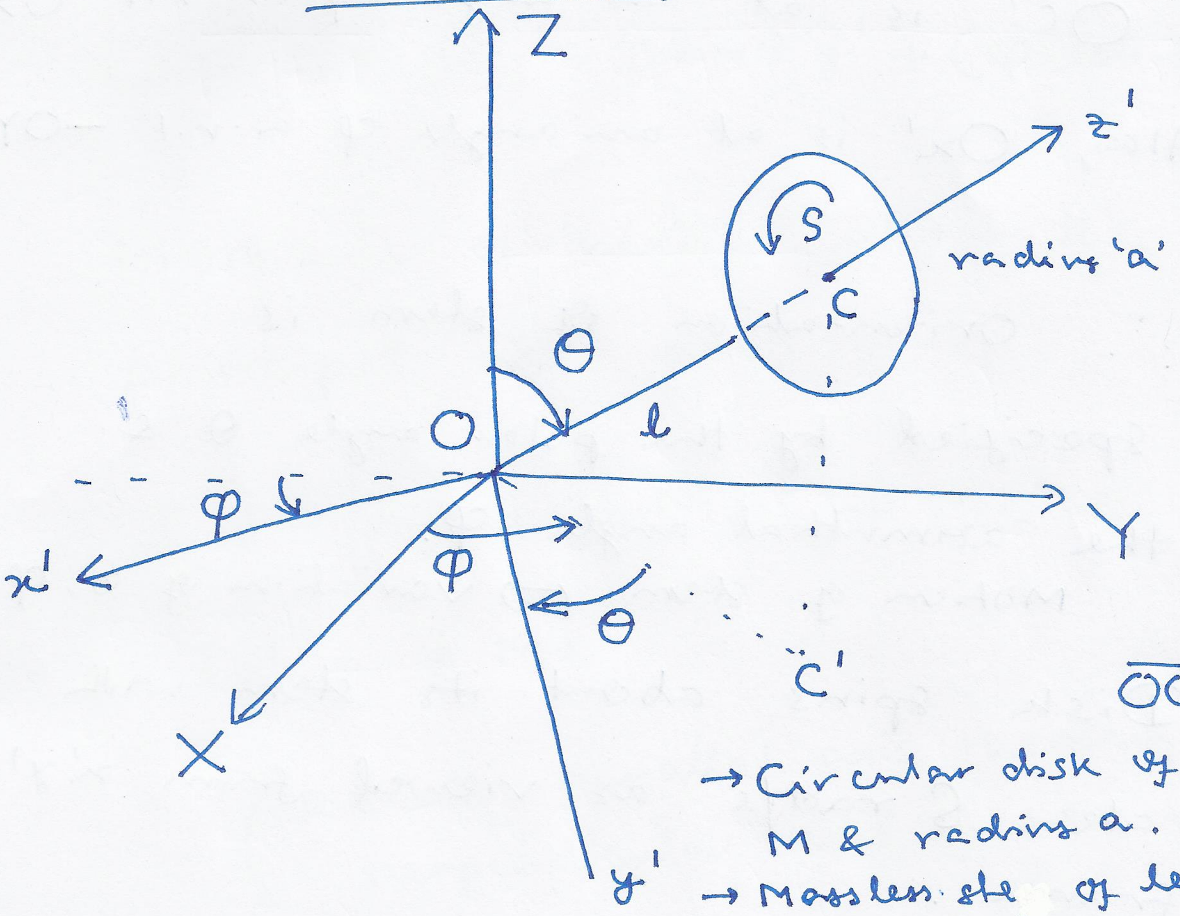
If I_1 is largest or smallest, $A > 0$ & motion is simple harmonic.
(i.e., $\exists$ restoring force)
ω_2 oscillates at freq. $\sqrt{A}$ with bounded amplitude.

If I_1 is intermediate, $A < 0$.
ω_2, ω_3 increase exponentially with time.

Spinning Top

or,

Gyroscope



$\overline{OC} = l$.

→ Circular disk of mass M & radius a.

→ Massless ste of length l.

Inertial frame XYZ

Rotating frame of principal axes $x'y'z'$.

The principal axes move with the stem of the disk.

Oz' along the stem

Ox' is always in the horizontal XY plane.

Oy' inclines below the XY plane by angle θ.

Oz' inclines at angle θ w.r.t. OZ.

- Projection of C.O.M. upon the XY plane falls at C′.

  OC′ is at an angle φ w.r.t OX.

  Also, Ox′ is at an angle φ w.r.t −OY axis.

  ∴ Orientation of stem is specified by the polar angle θ & the azimuthal angle φ.
- Motion of stem ⟺ variation of θ, φ.
- Disk spins about its stem with rate S rad/s as viewed from $x'y'z'$ frame.

In general, the total angular velocity of top will also involve variations in θ & φ.

∴ Total angular velocity vector

$$\vec{\omega} = -\dot{\theta}\,\hat{x}' + \dot{\varphi}\,\hat{Z} + S\,\hat{z}'.$$

But $\hat{Z} = -\sin\theta\,\hat{y}' + \cos\theta\,\hat{z}'$.

$\therefore$ In terms of principal axes components,

$$\vec{\omega} = -\dot{\theta}\,\hat{x}' - \dot{\phi}\sin\theta\,\hat{y}' + (\dot{\phi}\cos\theta + S)\,\hat{z}'.$$

Moments of inertia about the principal axes are,

$$I_{x'} = I_{y'} = \frac{1}{4}Ma^2 + Ml^2.$$

$$I_{z'} = \frac{1}{2}Ma^2$$

(Recall parallel axis theorem & perpendicular axis theorem.

As, $\vec{L} = I_{x'}\omega_{x'}\,\hat{x}' + I_{y'}\omega_{y'}\,\hat{y}' + I_{z'}\omega_{z'}\,\hat{z}'.$

$$\Rightarrow \vec{L} = \left(\frac{1}{4}Ma^2 + Ml^2\right)(-\dot{\theta}\,\hat{x}' - \dot{\phi}\sin\theta\,\hat{y}') + \frac{1}{2}Ma^2(\dot{\phi}\cos\theta + S)\,\hat{z}'$$

Special case : Steady precession at angle θ.

$\dot{\theta} = 0$

$\dot{\varphi}$ = constant

S = constant

Also, assume $S >> \dot{\varphi}$.

$\therefore \vec{L} = \frac{1}{2} M a^2 S \hat{z}'$.

Also, angular velocity of coordinate axis which do not spin with motion $S\hat{z}'$ (with $\dot{\theta} = 0$) is,

$$\vec{\omega}' = \dot{\varphi} \hat{Z}.$$

$$\therefore \frac{d\vec{L}}{dt} = \vec{\omega}' \times \vec{L} \quad (\because \dot{\varphi} \text{ \& } S \text{ are constants.})$$

$$= \dot{\varphi} \hat{Z} \times \frac{1}{2} M a^2 S \hat{z}'$$

$$= \frac{1}{2} M a^2 \dot{\varphi} S (\hat{Z} \times \hat{z}')$$

$$= -\frac{1}{2} M a^2 S \dot{\varphi} \sin\theta \, \hat{x}'.$$

But torque due to gravity

$$\vec{\tau} = -Mgl \sin\theta \, \hat{x}'.$$

Equating, $\Rightarrow$

$$-\frac{1}{2} Ma^2 S \dot{\varphi} \sin\theta = Mgl \sin\theta.$$

$$\therefore \quad \dot{\varphi} = \frac{Mgl}{\frac{1}{2} Ma^2 S}.$$

In general $\quad \dot{\varphi} = \dfrac{Mgl}{I_{z'} S}$

provided $S \gg \dot{\varphi}$ and steady precession.

+ DEMO

Check: For $S \gg \dot{\varphi}$,

$$\vec{L} = I_{z'} S \hat{z}'.$$

$$\frac{d\vec{L}}{dt} = \dot{\varphi} \hat{Z} \times I_{z'} S \hat{z}' = -I_{z'} S \dot{\varphi} \sin\theta \, \hat{x}'.$$

We dealt with very special case.
The formalism is crucial to developing technologies including inertial navigation & gyroscopic stabilization.