Central Forces

$$\vec{F} = m_{eff}\vec{a} = m_{eff}\left[(\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}\right].$$

Central force

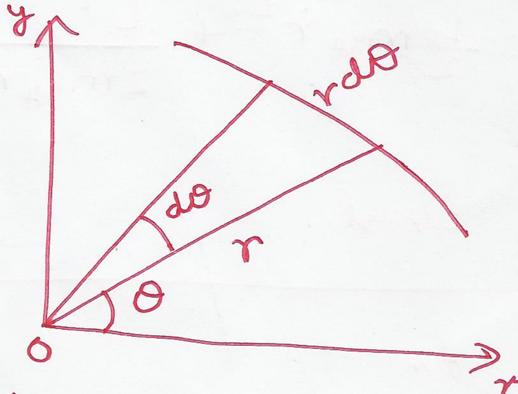
$$\vec{F} = \vec{F}(\vec{r}) = \underbrace{\left(\frac{k}{r^2}\right)}_{f(\vec{r})}\hat{r}.$$

$$\Rightarrow m_{eff}\ddot{r} - m_{eff}r\dot{\theta}^2 = \frac{k}{r^2}. \quad \text{—(i)}$$

$$\& \quad m_{eff}r\ddot{\theta} + 2m_{eff}\dot{r}\dot{\theta} = 0. \quad \text{—(ii)}$$

Eq.(ii) $\Rightarrow \dfrac{d}{dt}(m_{eff}r^2\dot{\theta}) = 0.$

$\therefore \quad m_{eff}r^2\dot{\theta} = \text{const} = L$ (ang. mom.).



Area swept by the radius vector in time dt

is $dA = \frac{1}{2} r . r d\theta = \frac{r^2 d\theta}{2}$

$\therefore$ Areal velocity $= \frac{1}{2}r^2\dot{\theta} = \frac{L}{2m_{eff}} =$ constant.

Also,

$$E = T + V$$

$$= \frac{1}{2} m_{eff} \dot{r}^2 + \frac{1}{2} m_{eff} r^2 \dot{\theta}^2 + V(r).$$

$$= \frac{1}{2} m_{eff} \dot{r}^2 + \frac{L^2}{2 m_{eff} r^2} + V(r).$$

We had from (ii),

$$m_{eff} \ddot{r} = -\frac{d}{dr}\left(V + \frac{L^2}{2 m_{eff} r^2}\right).$$

Multiply both sides by $\dot{r}$

& use $\dot{r}\frac{d}{dr} = \frac{d}{dt}$

$$\Rightarrow m_{eff} \dot{r} \ddot{r} = -\frac{d}{dt}\left(V + \frac{L^2}{2 m_{eff} r^2}\right).$$

or, $$\frac{d}{dt}\left[\frac{1}{2} m_{eff} \dot{r}^2 + \frac{L^2}{2 m_{eff} r^2} + V\right] = 0.$$

$$\therefore \frac{1}{2} m_{eff} \dot{r}^2 + \frac{L^2}{2 m_{eff} r^2} + V = \text{const.}$$

$$E = \text{const.}$$

$$\Rightarrow \boxed{\int_{r_0}^{r} \frac{dr}{\left[\frac{2}{m_{eff}}\left(E - V - \frac{L^2}{2 m_{eff} r^2}\right)\right]^{1/2}} = \int_0^t dt = t.}$$

Also, $\dot{\theta} = \dfrac{L}{m_{eff} r^2}$.

$$\boxed{\Rightarrow \int_{\theta_0}^{\theta} d\theta = \int_0^t \frac{L}{m_{eff} r^2}\, dt}.$$

Also, $d\theta = \dfrac{d\theta}{dt}\dfrac{dt}{dr}\, dr = \dfrac{\dot{\theta}}{\dot{r}}\, dr$

$$= \frac{L}{m_{eff} r^2 \dot{r}}\, dr.$$

$$\therefore \quad d\theta = \frac{(L/r^2)}{m_{eff}\dot{r}}\, dr.$$

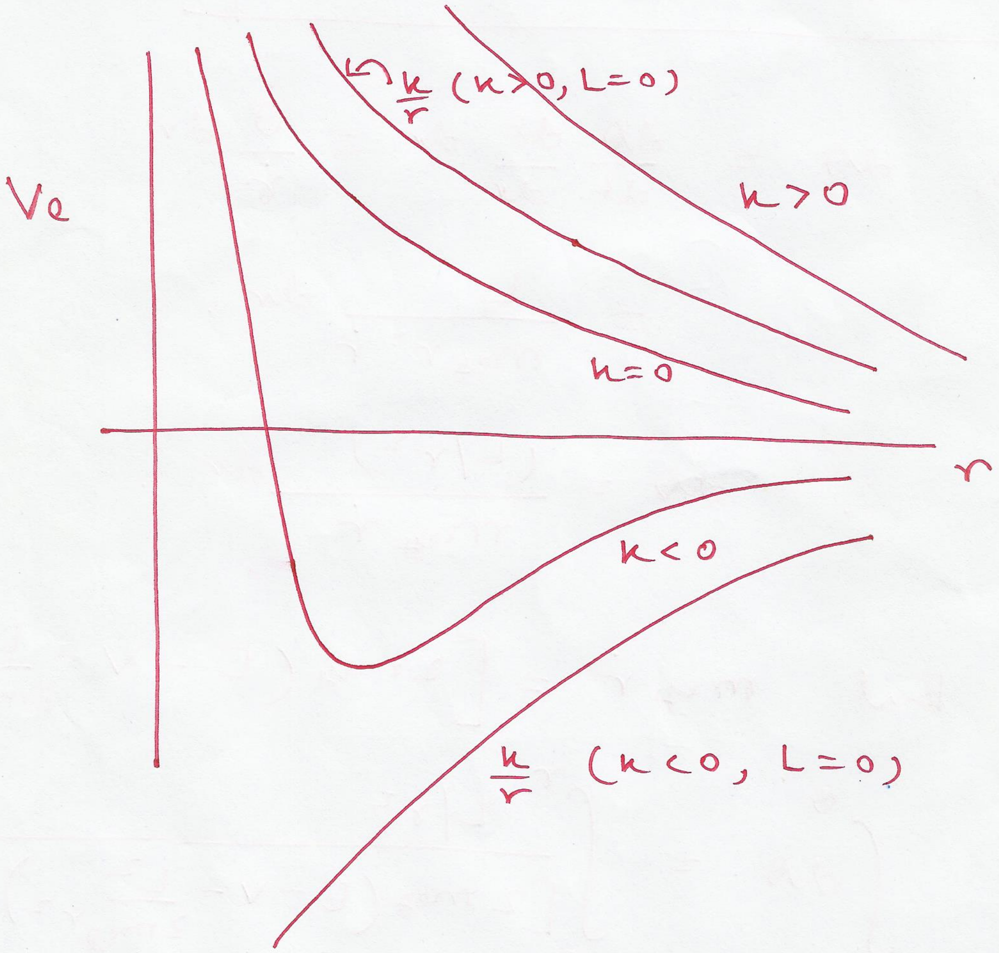
But $m_{eff}\dot{r} = \left[2 m_{eff}\left(E - V - \dfrac{L^2}{2 m_{eff} r^2}\right)\right]^{1/2}$.

$$\Rightarrow \int_{\theta_0}^{\theta} d\theta = \int_{r_0}^{r} \frac{L/r^2}{\left[2 m_{eff}\left(E - V - \frac{L^2}{2 m_{eff} r^2}\right)\right]^{1/2}}.$$

$$F(r) = \frac{k}{r^2}.$$

$$\therefore V(r) = + \frac{k}{r}$$

$$V_e = \frac{k}{r} + \frac{L^2}{2 m_{eff} r^2}.$$



$$m_{eff}\ddot{r} = F(r) + \frac{L^2}{m_{eff} r^3} \quad \text{——} \quad (A)$$

Let $u = \frac{1}{r}$

$$\frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta}$$

$$= -\frac{1}{r^2}\frac{dr}{dt}\frac{dt}{d\theta}.$$

$$\therefore \frac{du}{d\theta} = -\frac{\dot{r}}{r^2\dot{\theta}} = -\frac{m_{eff}}{L}\dot{r}.$$

$$\frac{d^2u}{d\theta^2} = \frac{d}{d\theta}\left(\frac{du}{d\theta}\right)$$

$$= \frac{d}{d\theta}\left(-\frac{m_{eff}}{L}\dot{r}\right)$$

$$= \frac{d}{dt}\left(-\frac{m_{eff}}{L}\dot{r}\right)\frac{dt}{d\theta}$$

$$= -\frac{m_{eff}}{L}\frac{\ddot{r}}{\dot{\theta}}$$

$$= -\frac{m_{eff}\, r^2}{L^2}\ddot{r}.$$

$$\therefore \frac{d^2u}{d\theta^2} + u = -\frac{m_{eff}}{L^2u^2}F\left(\frac{1}{u}\right).$$

$$F(r) = \frac{|k|}{r^2}.$$

$$F\left(\frac{1}{u}\right) = |k| u^2.$$

$$\therefore \quad \frac{d^2u}{d\theta^2} + u = \frac{|k| m_{eff}}{L^2}.$$

Let $\quad y = u - \frac{|k| m_{eff}}{L^2}$

$$\Rightarrow \quad \frac{d^2y}{d\theta} + y = 0.$$

$$y = A\cos(\theta - \theta_0).$$

$$\therefore \quad \frac{1}{r} = \frac{|k| m_{eff}}{L^2} + A\cos(\theta - \theta_0)$$

or, $$\frac{L^2/|k| m_{eff}}{r} = 1 + \frac{L^2 A}{|k| m_{eff}} \cos(\theta - \theta_0).$$

$$\frac{l}{r} = 1 + \epsilon\cos\theta.$$

$$l = \frac{L^2}{|k|\mu}; \quad \epsilon = \frac{L^2 A}{|k|\mu}.$$

Extremal values of $\cos\theta = \pm 1$.

$$\left.\begin{aligned} \frac{1}{r_1} &= \frac{m|k|}{L^2} + A, \\ \frac{1}{r_2} &= \frac{m|k|}{L^2} - A. \end{aligned}\right\} \text{turning points.}$$

Turning points are also roots of the equation,

$$E - V_e(\vec{r}) = E + \frac{|k|}{r} - \frac{L^2}{2mr^2} = 0.$$

The roots are,

$$\frac{1}{r_{1,2}} = \frac{m|k|}{L^2} \pm \frac{m|k|}{L^2}\sqrt{1 + \frac{2EL^2}{mk^2}}.$$

$$\therefore \frac{1}{r_1} - \frac{1}{r_2} = 2A = \frac{2m|k|}{L^2}\sqrt{1 + \frac{2EL^2}{mk^2}}.$$

$$\therefore A = \frac{m|k|}{L^2}\sqrt{1 + \frac{2EL^2}{mk^2}}.$$

$\Rightarrow$ Eccentricity, ϵ

$$= \frac{L^2 A}{m|k|} = \frac{L^2}{m|k|}\frac{m|k|}{L^2}\sqrt{1 + \frac{2EL^2}{mk^2}}.$$

$$= \sqrt{1 + \frac{2EL^2}{mk^2}}.$$

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$\epsilon = 0$	Circle	$E = -\frac{mk^2}{2L^2}$
$\epsilon = 1$	parabola	$E = 0.$
$\epsilon < 1$	Ellipse	$E < 0, \neq -\frac{mk^2}{2L^2}$
$\epsilon > 1$	Hyperbola	$E > 0$