Additional conserved quantity

Problem statement

$$V(\vec r) = -\frac{k}{r}, \qquad k > 0$$

Show that Laplace - Runge - Lenz vector given by,

$$\vec W = \frac{1}{mk}(\vec L \times \vec p) + \frac{\vec r}{r}$$

is a conserved quantity

$$\dot{\vec W} = \frac{1}{mk}(\dot{\vec L} \times \vec p) + \frac{1}{mk}(\vec L \times \dot{\vec p}) + \frac{\dot{\vec r}}{r} - \frac{\dot r \vec r}{r^2}.$$

$$\dot{\vec L} = 0$$

$$\vec L = m(\vec r \times \dot{\vec r})$$

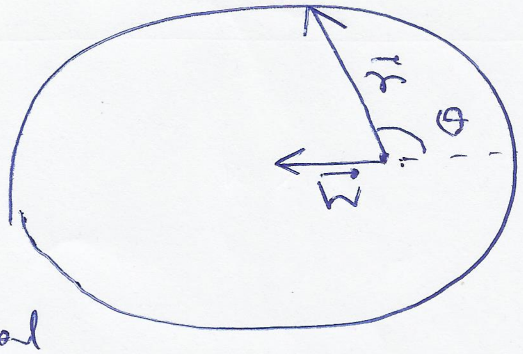
$$\dot{\vec p} = m\dot{\vec v} = -k\frac{\vec r}{r^3}$$

$$\Rightarrow \dot{\vec W} = \frac{1}{k}(\vec r \times \dot{\vec r}) \times \left(-\frac{k}{r^3}\vec r\right) + \frac{\dot{\vec r}}{r} - \frac{\dot r}{r^2}\vec r.$$

$$= \frac{1}{r^3}\left[-(\vec r \times \dot{\vec r}) \times \vec r + r^2 \dot{\vec r} - (\dot{\vec r} \cdot \vec r)\vec r\right]$$

$$= \frac{1}{r^3}\left[(\vec r \cdot \dot{\vec r})\vec r - (\vec r \cdot \vec r)\dot{\vec r} + r^2 \dot{\vec r} - (\dot{\vec r} \cdot \vec r)\vec r\right].$$

$= 0. \quad \Rightarrow \vec W$ is constant.



Both $\vec{L}\times\vec{p}$ and $\vec{r}$ lies within the orbital plane.

We next obtain the angle ψ between $\vec{W}$ & $\vec{r}$.

$$\vec{W}\cdot\vec{r} = |\vec{W}||\vec{r}|\cos\psi.$$

$$= \frac{1}{mk}(\vec{L}\times\vec{p})\cdot\vec{r} + \frac{\vec{r}\cdot\vec{r}}{r}.$$

$$= -\frac{1}{mk}\vec{L}\cdot(\vec{r}\times\vec{p}) + r.$$

$$= -\frac{L^2}{mk} + r.$$

$$\Rightarrow r = \frac{L^2/mk}{1 - W\cos\psi}.$$

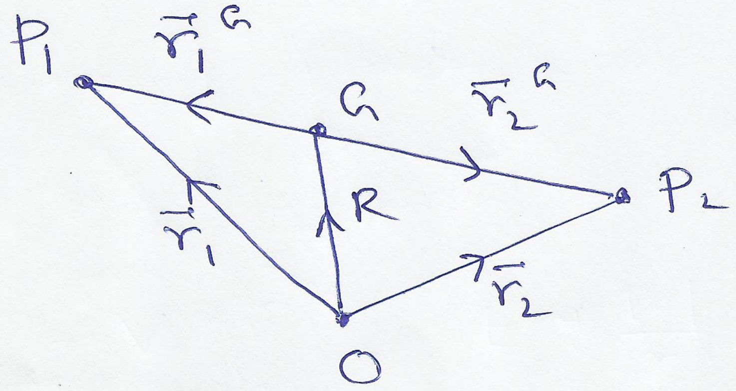
or, $$\frac{1}{r} = \frac{mk}{L^2} - \frac{W\cos\psi}{L^2/mk}.$$

$$= \frac{1}{(L^2/mk)}(1 - W\cos\psi).$$

$\therefore$ $W \leftrightarrow e$ & $\psi \leftrightarrow \theta + \pi$.

TWO BODY PROBLEM

Let P_1 & P_2 are moving under their mutual interaction.



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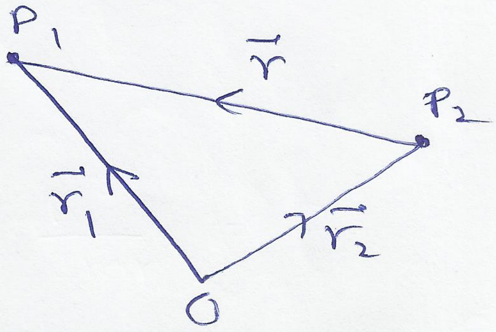


LAB

$$\vec{F}_1 = F(\vec{r})\,\hat{r}.$$

$$\vec{F}_2 = -F(\vec{r})\,\hat{r}.$$

where, $\vec{r} = \vec{r}_1 - \vec{r}_2$

& $\hat{r} = \dfrac{\vec{r}}{|\vec{r}|}$.

$$\Rightarrow m_1 \ddot{\vec{r}}_1 = F(\vec{r})\,\hat{r}.$$

$$m_2 \ddot{\vec{r}}_2 = -F(r)\,\hat{r}.$$

$\boxed{\mu = \left(\dfrac{m_1 m_2}{m_1+m_2}\right) \text{ is reduced mass.}}$

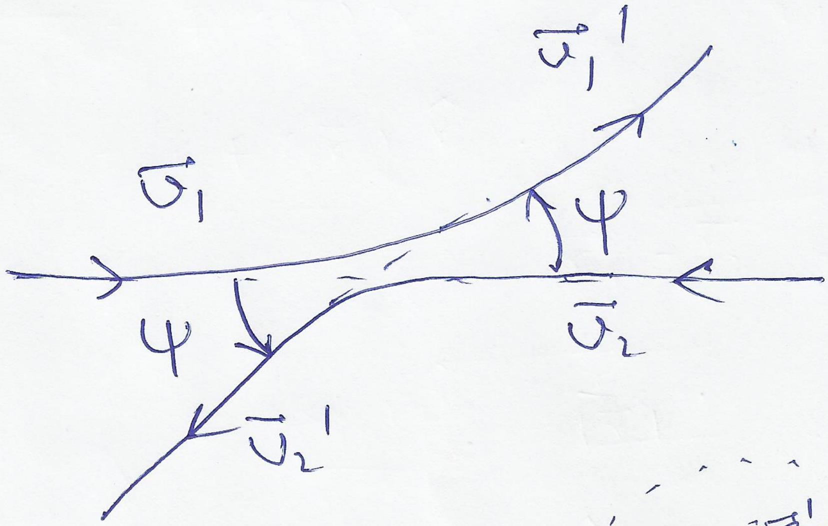
$$\Rightarrow \ddot{\vec{r}}_1 - \ddot{\vec{r}}_2 = F(r)\,\hat{r}\left(\frac{1}{m_1} + \frac{1}{m_2}\right).$$

$$\Rightarrow \left(\frac{m_1 m_2}{m_1+m_2}\right)\ddot{\vec{r}} = F(r)\,\hat{r}.$$

& $\vec{r}_1^{\,G} = \left(\frac{m_2}{m_1+m_2}\right)\vec{r}$.

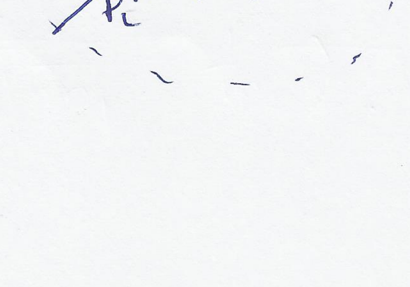
$\vec{r}_2^{\,G} = -\left(\frac{m_1}{m_1+m_2}\right)\vec{r}$.

Notion of zero momentum frame / center of mass frame.



$\vec{P}_1 + \vec{P}_2 = 0$.

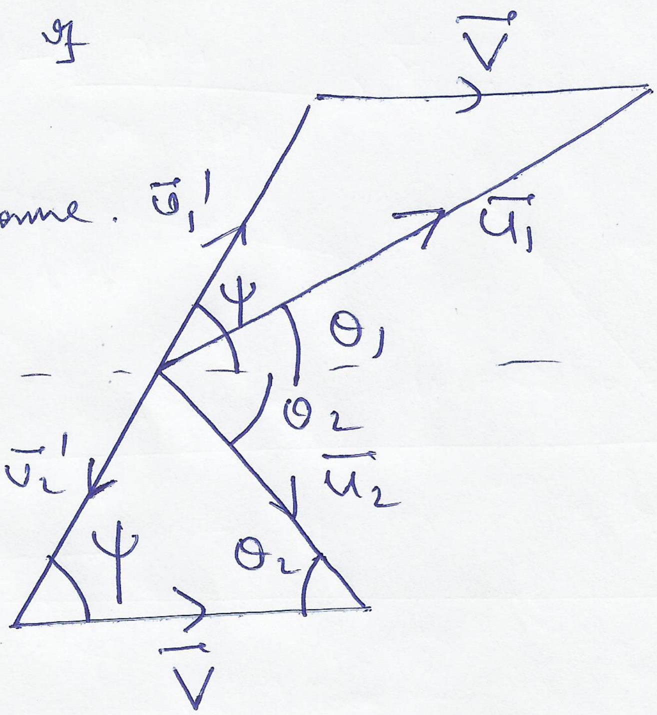
$\vec{P}_1' + \vec{P}_2' = 0$.



Elastic Collision

Let the final velocities in lab frame be $\vec{u}_1$ & $\vec{u}_2$

$\vec{V}$ is velocity of ZM frame w.r.t. lab frame.



$$\vec{u}_1 = \vec{v}_1{}' + \vec{V}$$

$$\vec{u}_2 = \vec{v}_2{}' + \vec{V}.$$

Show

$$\tan\theta_1 = \frac{\sin\psi}{\cos\psi + (m_1/m_2)}.$$