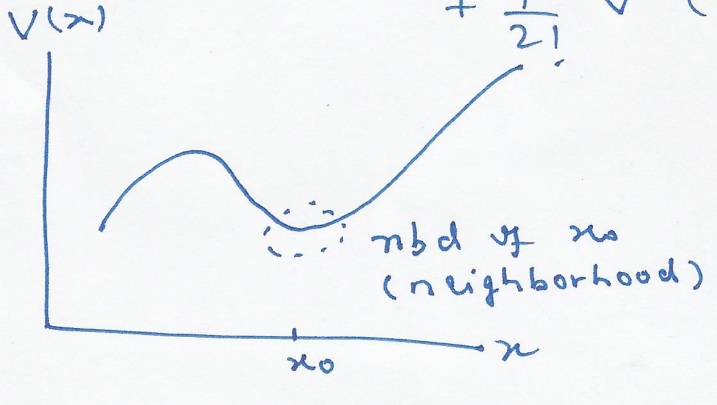
Small oscillations & the potential energy function

$$V(x) = V(x_0) + V'(x_0)(x-x_0) + \frac{1}{2!}V''(x_0)(x-x_0)^2 + \dots + \frac{1}{(n-1)!}V^{n-1}(x_0)(x-x_0)^{n-1} + R_n.$$

Here, x_0 represents the minimum of the potential.

For small oscillations, (x in the nbd of x_0),

$$V(x) \approx V(x_0) + V'(x_0)(x-x_0) + \frac{1}{2!}V''(x_0)(x-x_0)^2.$$



At $x = x_0$, potential has a minimum,

$$\Rightarrow V'(x_0) = 0.$$

$$\Rightarrow V(x_0) = \text{constant}.$$

Redefine zero of potential s.t.,

$$V(x_0) \equiv 0.$$

$$\therefore V(x) \sim \frac{1}{2} V''(x_0)(x - x_0)^2.$$

$$\equiv \frac{1}{2} K_{eff}(x - x_0)^2.$$

$$\therefore \omega_0 = \sqrt{\frac{K_{eff}}{m}} = \sqrt{\frac{V''(x_0)}{m}}.$$

Ex.

Consider a two atom molecule held in a plane by a potential

$$V(x) = \frac{A}{x^3} - \frac{B}{x^2}.$$

$$(A = u_0 a_0^3, \; B = u_0 a_0^2).$$

u_0 = unit of molecular energy.

a_0 = unit of molecular distance.

(a) Obtain bond length (if one of the atoms is at origin)

(b) Find molecular freq. of vibration?

(a.) At $x = x_b$, $V'(x) = 0$.

$$\Rightarrow \left(-\frac{3A}{x^4} + \frac{2B}{x^3}\right)\Bigg|_{x = x_b} = 0.$$

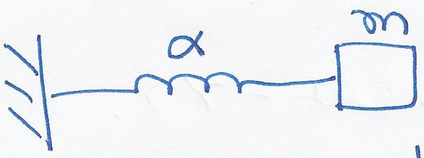
$$\therefore \; x_b = \frac{3A}{2B} = \frac{3}{2} a_0.$$

(b.) $$K_{eff} = V''(x_b) = \frac{12A}{x_b^5} - \frac{6B}{x_b^4}.$$

$$\therefore \; \omega = \sqrt{\frac{1}{m}\left(\frac{12 u_0 a_0^3}{(\frac{3}{2})^5 a_0^5} - \frac{6 u_0 a_0^2}{(\frac{3}{2})^4 a_0^4}\right)}$$

$$= \sqrt{\frac{1}{m} \frac{32 \times 4 u_0}{3^4 a_0^2} - \frac{32 \times 3 u_0}{3^4 a_0^2}}.$$

$$= \sqrt{\frac{32 u_0}{81 m a_0^2}}.$$



$$m\frac{d^2x}{dt^2} = \underbrace{-\alpha x}_{\substack{\text{restoring force} \\ \text{(Linear string)}}} \overbrace{-\beta\frac{dx}{dt}}^{\text{damping force}} + \underbrace{G(t)}_{\text{driving force}}.$$

$$\therefore \boxed{\frac{d^2x}{dt^2} + 2K\frac{dx}{dt} + \Omega^2 x = F(t)}$$

$$\alpha = m\Omega$$

$$\beta = 2mK.$$

(I.) $\underline{K = 0, F(t) = 0}$

$$\frac{d^2x}{dt^2} + \Omega^2 x = 0.$$

trial solⁿ. $x(t) = e^{\lambda t}$.

$$\Rightarrow \lambda^2 + \Omega^2 = 0.$$

$$\therefore \lambda = \pm i\Omega.$$

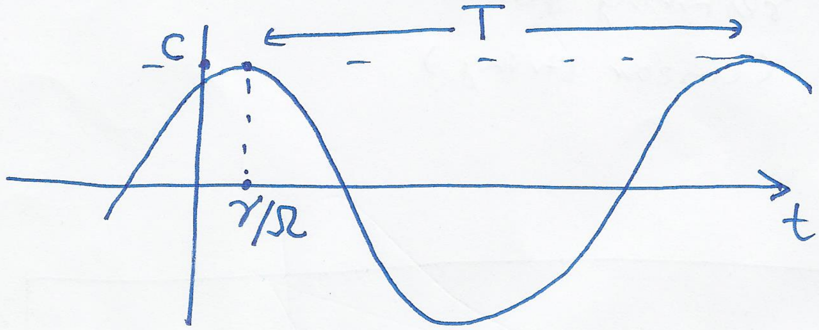
$$\Rightarrow x(t) = \underset{\substack{\downarrow \\ \frac{C-iD}{2}}}{A} e^{i\Omega t} + \underset{\substack{\searrow \\ \frac{C+iD}{2}}}{B} e^{-i\Omega t}.$$

$$\therefore x(t) = C\cos\Omega t + D\sin\Omega t.$$

or, alternatively,

$$x = C\cos(\Omega t - \gamma).$$

C & γ are real arbitrary constants with $C > 0$.



$T = \frac{2\pi}{\Omega}$

Ω = ang. freq. of osc.

(II.) $\underline{F(t) = 0}$

$$\frac{d^2x}{dt^2} + 2K\frac{dx}{dt} + \Omega^2 x = 0.$$

Trial: $x(t) = e^{\lambda t}$.

$$\Downarrow$$

$$\lambda^2 + 2K\lambda + \Omega^2 = 0.$$

i.e., $(\lambda + K)^2 = K^2 - \Omega^2$

$\exists$ three different cases depending on whether $K < \Omega$, $K = \Omega$, $K > \Omega$.

A: $K < \Omega$ Under damped

In this case

$$(\lambda + K)^2 = -\Omega_D^2$$

s.t., $\Omega_D^2 = \Omega^2 - K^2$.

i.e., $\Omega_D = \sqrt{\Omega^2 - K^2}$ is a positive real number.

$$\lambda = \begin{cases} -K + i\Omega_D . \\ -K - i\Omega_D . \end{cases}$$

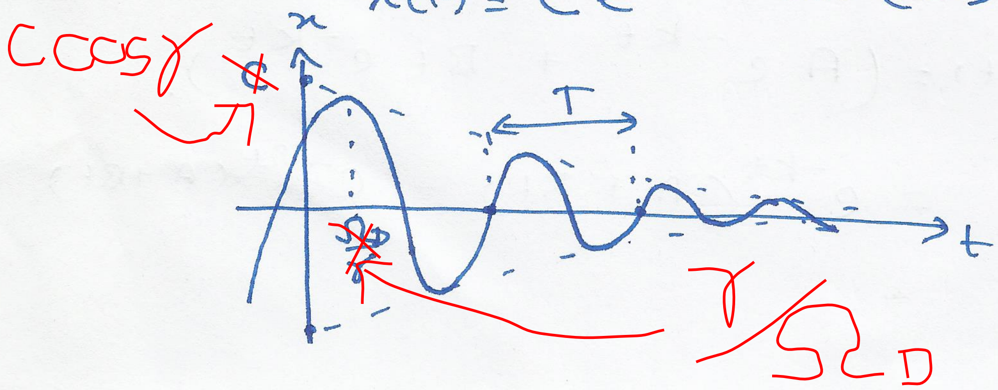
$$\therefore \quad x(t) = C_1 e^{-Kt + i\Omega_D t} + C_2 e^{-Kt - i\Omega_D t}$$

It can therefore be cast in the form,

$$x(t) = e^{-Kt}\left(A \cos \Omega_D t + B \sin \Omega_D t\right).$$

or, alternatively,

$$x(t) = C e^{-Kt} \cos(\Omega_D t - \gamma).$$



B. $K > \Omega$ Over damped

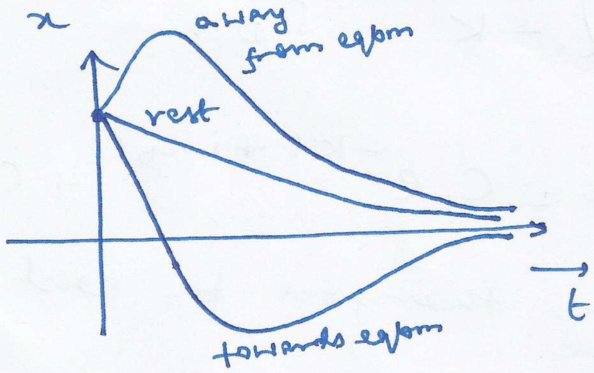
$\therefore (\lambda + K)^2 = \delta^2$

s.t., $\delta = (K^2 - \Omega^2)^{1/2}$ is a positive real number.

$$\lambda = \begin{cases} -K + \delta \\ -K - \delta \end{cases}$$

$$\therefore x(t) = e^{-Kt}(A e^{\delta t} + B e^{-\delta t}).$$

A, B are real arbitrary consts.



C. $K = \Omega$ Critically damped

$\therefore (\lambda + K)^2 = 0.$

$\Rightarrow \lambda = -K, -K$ (repeated roots).

$$\therefore x(t) = (A e^{-Kt} + B t e^{-Kt}).$$

$$= e^{-Kt}(A + Bt) = e^{-\Omega t}(A + Bt).$$

(III.) General case (Forced & Damped)

$$\frac{d^2x}{dt^2} + 2K\frac{dx}{dt} + \Omega^2 x = F_0 e^{ipt}.$$

$$x = \underbrace{x^{CF}} + a\cos(pt - \gamma).$$

same as sol^n for (II.)
(incl. diff. cases)

$$a = \frac{F_0}{\left((\Omega^2 - p^2)^2 + 4K^2p^2\right)^{1/2}}$$

$$\tan\gamma = \frac{2Kp}{\Omega^2 - p^2}.$$