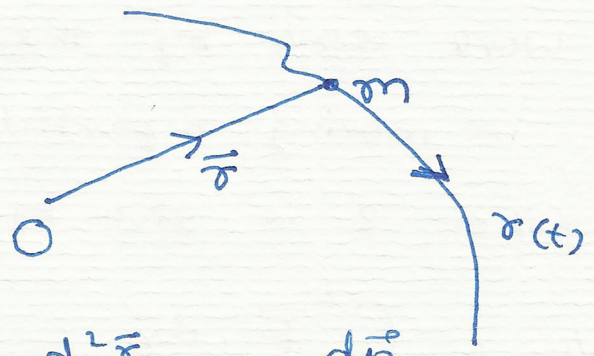
Mechanics of a particle



$$\vec{F} = m\frac{d^2\vec{r}}{dt^2} = \frac{d\vec{p}}{dt},$$

where $\vec{F}$ is the force acting on m.

Soln. of the eqr. is of the form $\vec{r} = \vec{r}(t)$.

(Determines the trajectory of mass m)

If no external force is acting,

$$\vec{F} = \frac{d\vec{p}}{dt} = 0$$

$$\Rightarrow \vec{p} = m\vec{v} = \text{const.}$$

Angular momentum about point O is defined as,

$$\vec{L} = \vec{r} \times \vec{p}$$

$$\therefore \vec{\tau} = \frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p}) = \dot{\vec{r}} \times \vec{p} + \vec{r} \times \dot{\vec{p}}.$$

But $\dot{\vec{r}} \times \vec{p} = \dot{\vec{r}} \times m\dot{\vec{r}} = 0.$

& $\vec{r} \times \dot{\vec{p}} = \vec{r} \times \vec{F}.$

$$\therefore \boxed{\vec{\tau} = \frac{d\vec{L}}{dt} = \vec{r} \times \vec{F}}$$

If $\vec{\tau} = 0$, $\vec{L}$ = a constant.

(e.g., planets moving around sun).

The work done by the total external force in moving a particle from position 1 to position 2 is given by,

$$W_{12} = \int_1^2 \vec{F} \cdot d\vec{r}$$

$$= \int_1^2 m \frac{d\vec{v}}{dt} \cdot d\vec{r} \quad \text{if } m \text{ is constant}$$

$$= \int_1^2 m \frac{d\vec{v}}{dt} \cdot \left(\frac{d\vec{r}}{dt}\right) dt$$

$$= \frac{1}{2} m \int_1^2 d(v^2).$$

$$= \frac{1}{2} m (v_2^2 - v_1^2).$$

$$= T_2 - T_1.$$ (kinetic energies of the particle at positions 2 & 1.)

If $T_1 > T_2$, $W_{12} < 0$

work is done by the particle against the force & its K.E. has reduced.

If $T_1 < T_2$,

work is done by the force on the particle & its K.E. increases.

If the force field is such that work done along a closed path is 0 then the force is said to be "conservative".

i.e., if $\oint \vec{F}\cdot d\vec{r} = 0$, then $\vec{F}$ is conservative.

$$\oint \vec{F}\cdot d\vec{r} = \int_\sigma (\nabla\times\vec{F})\cdot d\vec{\sigma}.$$

$\therefore$ for a conservative force field,

$$\nabla\times\vec{F} = 0.$$

$$(\because\ \nabla\times(\nabla\phi) = 0.)$$

$$\Rightarrow \vec{F} = -\nabla\phi.$$

ϕ → potential energy of the particle.

$$\therefore W_{12} = \int_1^2 \vec{F}\cdot d\vec{r} = -\int_1^2 \nabla\phi\cdot d\vec{r}.$$

$$= -\int_1^2 \left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz\right).$$

$$= -\int_1^2 d\phi = \phi_1 - \phi_2.$$

But $W_{12} = T_2 - T_1$! $\Rightarrow T_2 - T_1 = \phi_1 - \phi_2$.

$$\therefore \quad U = T_1 + \phi_1 = T_2 + \phi_2 = \text{const}.$$

Useful Identities

$$\vec{V} = \hat{e}_1 V_1 + \hat{e}_2 V_2 + \hat{e}_3 V_3 .$$

$$= \frac{\hat{e}_1}{h_2 h_3}(h_2 h_3 V_1) + \frac{\hat{e}_2}{h_1 h_3}(h_1 h_3 V_2) + \frac{\hat{e}_3}{h_1 h_2}(h_1 h_2 V_3).$$

$$\nabla \phi = \frac{\hat{e}_1}{h_1}\frac{\partial}{\partial u_1} + \frac{\hat{e}_2}{h_2}\frac{\partial}{\partial u_2} + \frac{\hat{e}_3}{h_3}\frac{\partial}{\partial u_3}$$

$$\nabla \cdot \vec{V} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(h_2 h_3 V_1) + \frac{\partial}{\partial u_2}(h_3 h_1 V_2) + \frac{\partial}{\partial u_3}(h_1 h_2 V_3)\right].$$

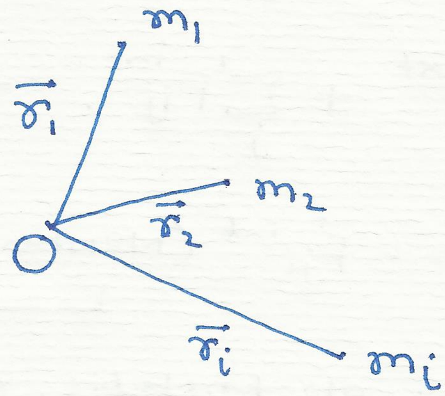
$$\nabla \times \vec{V} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1 \hat{e}_1 & h_2 \hat{e}_2 & h_3 \hat{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 V_1 & h_2 V_2 & h_3 V_3 \end{vmatrix}$$

$$\nabla^2 \phi = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_1 h_2 h_3}{h_1^2}\right)\frac{\partial}{\partial u_1} + \ldots\right].$$

cyclic terms.

Mechanics of a system of particles

System of N particles



Let $\vec{F}^{ext}$ be the total external force acting on the system of N particles with $\vec{F}_i^{ext}$ being the external force acting on i^{th} particle. Also, internal force $\sum_j \vec{F}_{ij}^{int}$ acts on i^{th} particle due to all the other particles.

i.e., $\vec{F}_i^{int} = \sum_j{}' \vec{F}_{ij}^{int}$ (prime ≡ excludes i^{th} particle) — (i)

Total mass of the system is given by $M = \sum_{i=1}^{N} m_i \equiv \sum_i m_i$. — (ii)

The center of mass of the system is defined as the point whose position vector is given by,

$$\vec{R}_{cm} = \frac{\sum_i m_i \vec{r}_i}{\sum_i m_i} = \frac{1}{M} \sum_i m_i \vec{r}_i . \quad \text{— (iii)}$$

~~Let~~

The equation (EOM) of motion of the i^{th} particle can be written as,

$$\dot{\vec{p}}_i = \vec{F}_i^{ext} + \sum_j{}' \vec{F}_{ij}^{int}. \quad \text{—— (iv)}$$

Now, $\vec{F}_{ij}^{int} = -\vec{F}_{ji}^{int}$ (Newton's 3rd law). —— (v)

(examples : electrostatic, gravitational)

Note :- EM force is a velocity dep. force & does not act along the line joining i^{th} & j^{th} particles.

$\therefore$ EOM of the system is

$$\sum_i \dot{\vec{p}}_i = \sum_i \vec{F}_i^{ext} + \sum_{ij}{}' \vec{F}_{ij}^{int}. \quad \text{—— (vi)}$$

or, $$\frac{d^2}{dt^2}\sum_i m_i \vec{r}_i = \sum_i \vec{F}_i^{ext} + \sum_{ij}{}' \vec{F}_{ij}^{int}. \quad \text{—— (vii)}$$

Consider $\sum_{ij}{}' \vec{F}_{ij}^{int}$.

i & j are dummy indices,

$$\Rightarrow \sum_{ij}{}' \vec{F}_{ij}^{int} = \sum_{ij}{}' \vec{F}_{ji}^{int}. \quad \text{—— (viii)}$$

Further, using (v) & (viii),

$$\sum_{ij}{}' \vec{F}_{ij}^{int} = \frac{1}{2}\left(\sum_{ij}{}' \vec{F}_{ij}^{int} + \sum_{ij}{}' \vec{F}_{ji}^{int}\right)$$

$$= \frac{1}{2}\sum_{ij}{}'\left(\vec{F}_{ij}^{int} + \vec{F}_{ji}^{int}\right) = 0 \quad \text{—— (ix)}$$

$$\therefore \quad \frac{d^2}{dt^2}\sum_i m_i \vec{r}_i = \sum_i \vec{F}_i^{ext} = \vec{F}^{ext}.$$

$$\Rightarrow \frac{d^2}{dt^2}(M\vec{R}_{cm}) = \vec{F}^{ext}.$$

$$\text{i.e.,} \quad M\frac{d^2\vec{R}_{cm}}{dt^2} = \vec{F}^{ext}. \qquad \text{—— (x.)}$$

$$\left(\because \frac{dM}{dt} = 0.\right).$$

$$\text{i.e.,} \quad \boxed{M\ddot{\vec{R}}_{cm} = \vec{F}^{ext}} \qquad \text{—— (xi.)}$$

Also, total linear momentum of the system is given by,

$$\vec{P} = \sum_i m_i \dot{\vec{r}}_i = \frac{d}{dt}\sum_i m_i \vec{r}_i$$

$$= \frac{d}{dt}(M\vec{R}_{cM}) = M\dot{\vec{R}}_{cM}. \qquad \text{—— (xii.)}$$

Using (xi.) & (xii.),

$$\boxed{\dot{\vec{P}} = \vec{F}^{ext}} \qquad \text{—— (xiii.)}$$

Center of mass behaves like a particle whose mass is equal to the total mass of the system & is acted upon by total external force $\vec{F}^{ext}$.

Corrolaries

(1.) This exist only for force fields for which $\vec{F}_{ij}^{\,int} = \vec{F}_{ji}^{\,int}$.

(2.) If total external force $\vec{F}^{\,ext} = 0$, then the total linear momentum of the system of N particles is conserved.

i.e., if $\vec{F}^{\,ext} = 0$, then $\dot{\vec{P}} = 0$ or, $\vec{P} = $ const.

Angular momentum of the system of particles

$\vec{l}_i = \vec{r}_i \times \vec{p}_i$ angular momentum of the i^{th} particle about some point.

$\therefore$ total angular momentum of the system

$$\vec{L} = \sum_i \vec{l}_i = \sum_i \vec{r}_i \times \vec{p}_i .$$

$\therefore$ Torque acting on the system is,

$$\vec{\tau} = \frac{d\vec{L}}{dt} = \frac{d}{dt} \sum_i \vec{r}_i \times \vec{p}_i .$$

$$= \sum_i \dot{\vec{r}}_i \times \vec{p}_i + \sum_i \vec{r}_i \times \dot{\vec{p}}_i .$$

But, $\sum_i \vec{r}_i \times \vec{p}_i = \sum_i \dot{\vec{r}}_i \times m_i \dot{\vec{r}}_i$

$$= \sum_i m_i \dot{\vec{r}}_i \times \dot{\vec{r}}_i = 0.$$

Also, $\sum_i \vec{r}_i \times \dot{\vec{p}}_i = \sum_i \vec{r}_i \times \left(\vec{F}_i^{ext} + \sum_j{}' F_{ij}^{int}\right).$

$$\therefore \quad = \sum_i \vec{r}_i \times \vec{F}_i^{ext}.$$

Because,

$$\sum_i \sum_j{}' \vec{r}_i \times \vec{F}_{ij}^{int} = \sum_j \sum_i{}' \vec{r}_j \times \vec{F}_{ji}^{int}.$$

$$\therefore \sum_{ij}{}' \vec{r}_i \times \vec{F}_{ij}^{int} = \frac{1}{2}\left(\sum_{ij}{}' \left\{ \begin{array}{l} \vec{r}_i \times \vec{F}_{ij}^{int} \\ + \vec{r}_j \times \vec{F}_{ji}^{int} \end{array} \right\}\right).$$

$$= \frac{1}{2} \sum_{ij}{}' (\vec{r}_i - \vec{r}_j) \times \vec{F}_{ij}^{int}.$$

But $\vec{F}_{ij}^{int}$ acts along $\vec{r}_i - \vec{r}_j$ for case under consideration.

$$\therefore \sum_{ij}{}' \vec{r}_i \times \vec{F}_{ij}^{int} = 0.$$

$$\therefore \vec{\tau} = \frac{d\vec{L}}{dt} = \sum_i \vec{r}_i \times \vec{F}_i^{ext} = \sum_i \vec{\tau}_i.$$

Total torque on the system is equal to the vector sum of the torques acting on the individual particles.

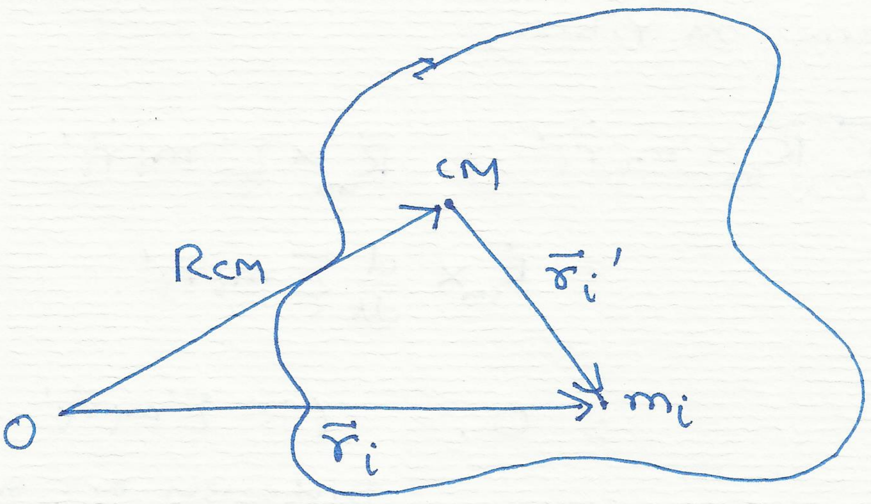
Note:

- Only valid if $\vec{F}_{ij}^{\,int}$ acts along $(\vec{r}_i - \vec{r}_j)$.

- If $\vec{\tau} = 0 \Rightarrow \frac{d\vec{L}}{dt} = 0$.

$$\text{i.e.,} \quad \vec{L} = \text{const.}$$

i.e., if total external torque acting on the system of N particles is zero, then the total angular momentum of the system is conserved.



m_i & $\vec{r}_i$ denote the mass and the position vector of the i^{th} particle with reference to point O.

Let $\vec{r}_i{}'$ be the position vector of the i^{th} particle with reference to the center of mass.

$$\therefore \quad \vec{r}_i = \vec{R}_{cm} + \vec{r}_i{}'.$$

$$\therefore \quad \dot{\vec{r}}_i = \dot{\vec{R}}_{cm} + \dot{\vec{r}}_i{}'.$$

⇒ Total angular momentum of the system of particles is,

$$L = \sum_i \vec{r}_i \times \vec{P}_i$$

$$= \sum_i (\vec{R}_{cm} + \vec{r}_i{}') \times m_i (\dot{\vec{R}}_{cm} + \dot{\vec{r}}_i{}')$$

$$= \sum_i m_i \vec{R}_{cm} \times \dot{\vec{R}}_{cm} + \sum_i \vec{R}_{cm} \times m_i \dot{\vec{r}}_i{}'$$
$$+ \sum_i m_i \vec{r}_i{}' \times \dot{\vec{R}} + \sum_i m_i \vec{r}_i{}' \times \dot{\vec{r}}_i{}'$$

2^{nd} term on r.h.s.,

$$\sum_i \vec{R}_{cm} \times m_i \dot{\vec{r}}_i' = \vec{R}_{cm} \times \sum_i m_i \dot{\vec{r}}_i'$$

$$= \vec{R}_{cm} \times \frac{d}{dt} \sum_i m_i \vec{r}_i'.$$

$$= 0. \quad (\because \sum m_i \vec{r}_i' = 0.)$$

Distances are measured w.r.t. CM.

3^{rd} term on r.h.s.,

$$\sum_i m_i \vec{r}_i' \times \dot{\vec{R}}_{cm} = \left(\sum_i m_i \vec{r}_i'\right) \times \dot{\vec{R}}_{cm}.$$

$$= 0. \quad (\text{as above.})$$

$$\therefore \vec{L} = \sum_i m_i \vec{R}_{cm} \times \dot{\vec{R}}_{cm} + \sum_i m_i \vec{r}_i' \times \dot{\vec{r}}_i'.$$

$$= \vec{R}_{cm} \times \dot{\vec{R}}_{cm} \sum_i m_i + \sum_i \vec{r}_i' \times m_i \dot{\vec{r}}_i'.$$

$$= \vec{R}_{cm} \times \dot{\vec{R}}_{cm} M + \sum_i \vec{r}_i' \times \vec{P}_i'.$$

i.e., $\vec{L} = \underbrace{(\vec{R}_{cm} \times \vec{P}_{cm})}_{\text{ang. mom. of the total mass concentrated at the center of mass about point O.}} + \underbrace{\vec{L}'}_{\text{ang. mom. of the system about its center of mass.}}$