Variable mass situation

(I.) Condensation of a water droplet

A dust particle of negligible mass at $t=0$ begins to fall under the influence of gravity through saturated water vapour. The vapor condenses with constant rate λ (gm-cm^{-1}) on the dust particle & forms a water droplet of steadily increasing mass. [neglect friction & collisions].

The external force acting on droplet is gravity:

$$F_g = mg.$$

$$\Rightarrow mg = \frac{dm}{dt}v + m\frac{dv}{dt}.$$

$$\text{But } \frac{dm}{dt} = \frac{dm}{dx}\cdot\frac{dx}{dt} = \lambda v.$$

$$\therefore mg = \lambda v^2 + m\frac{dv}{dt},$$

$$\Rightarrow a = \frac{dv}{dt} = \left(\frac{mg - \lambda v^2}{m}\right).$$

At $t=0$, $x=0$ & $m=0$.

$$\Rightarrow m = \lambda x.$$

$$\therefore \boxed{a = g - \frac{v^2}{x}.}$$

$$\Rightarrow x\cdot\frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^2 - gx = 0.$$

To solve this nonlinear differential equation, try ansatz $x = At^n$.

$$\Rightarrow (At^n)\,n(n-1)At^{n-2} + (nAt^{n-1})^2 - gAt^n = 0.$$

i.e., $\boxed{n(n-1)A^2t^{2n-2} + n^2A^2t^{2n-2} - gAt^n = 0}$

Demand $2n-2 = n$.

$$\Rightarrow n = 2.$$

Substituting in the boxed eq. above,

$$\Rightarrow 2A^2t^2 + 4A^2t^2 - gAt^2 = 0.$$

$$\therefore\ 6A^2 = gA.$$

$$\Rightarrow\ A = \frac{g}{6}.$$

$$\therefore\ \boxed{x = \frac{g}{6}t^2}$$

$$\therefore\ \boxed{a = \frac{d^2x}{dt^2} = \frac{g}{3}}$$

$\therefore$ Accl^n of the droplet is constant & independent of x, & equals $\frac{g}{3}$.

Other examples

- Rocket motion
- People travelling on an escalator
- Sand on a conveyor belt.
- Leaky bucket
- Chain released on a weight measuring scale.

(II.) Chain falling on a weighing scale

Length $= L$.

mass per unit length $= \sigma$.

Method - I

Let y = height of top of the chain.

+ve ↑

Let F be the desired force applied by the scale.

The net force on the entire chain is

$$F - (\sigma L) g.$$

Momentum of entire chain

$$= (\sigma y)\dot{y}. \quad \left(\begin{array}{c}\text{only} \\ \text{moving part} \\ \text{contributes}\end{array}\right)$$

$$\therefore \quad F - (\sigma L) g = \sigma y \ddot{y} + \sigma \dot{y}^2. \quad \text{— (I.)}$$

Part of chain which is still above the scale is in "free fall", i.e., $\ddot{y} = -g$.

Also conservation of energy

$$\Rightarrow \quad \sigma (L - y) g y = \tfrac{1}{2} \sigma y \dot{y}^2$$

$$\Rightarrow \dot{y} = \sqrt{2g(L-y)}\,.$$

putting in eq. (I),

$$\Rightarrow F = \sigma L g - \sigma y g + 2\sigma(L-y)g.$$

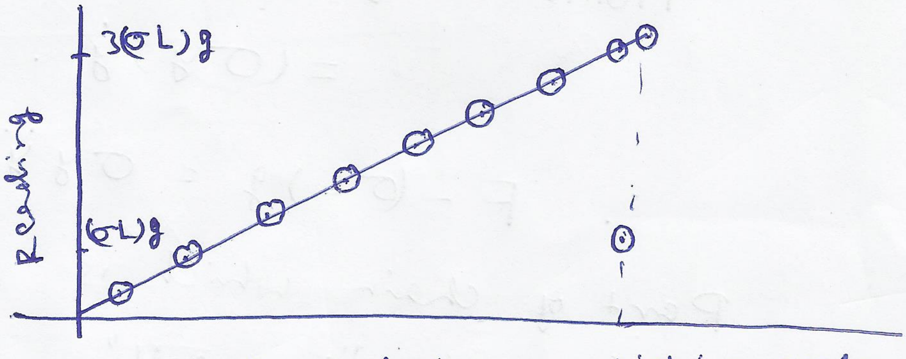
i.e., $F = 3\sigma(L-y)g.$

Three times the weight of the chain already on the scale.

<u>Check</u>: At $y = L$, $F = 0$.
(Entire chain hanging)

At $y = 0$, $F = 3(\sigma L)g$.
(just before the last bit falls on the scale)

Note: Once the chain is completely on the scale, reading drops down to $(\sigma L)g$.



(III.) Rockets

(a.) Free space

At $t = 0$, the ignition happens & the fuel products are ejected backwards with speed u relative to the rocket.

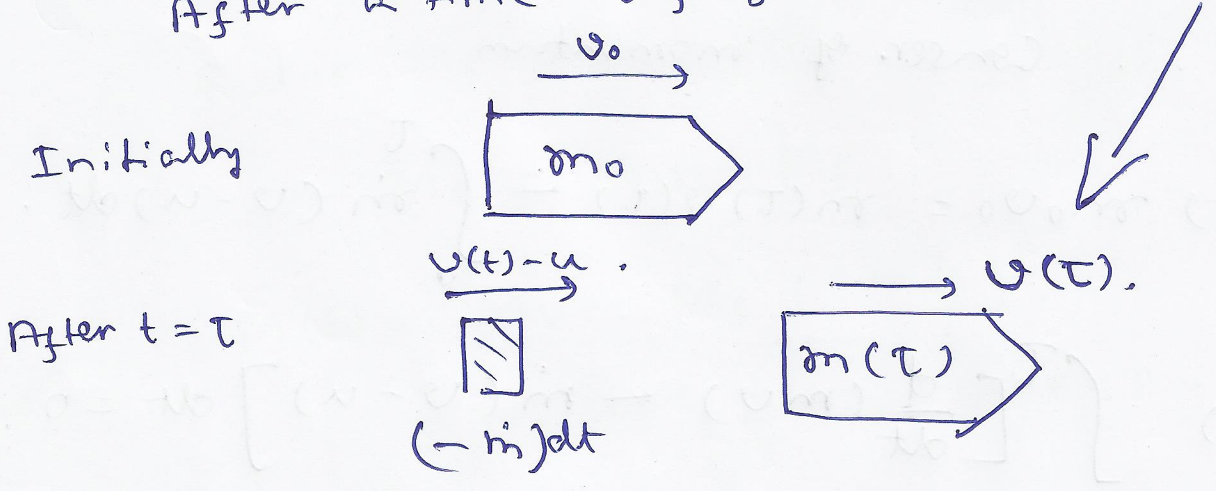
The fuel burn continues for a time T, at the end of which mass (rocket + remaining fuel) is m_1.

Let $m = m(t)$ be the (rocket + unburned fuel) mass at time t.

Rate of ejection of mass at time t is $-\dot{m}$.

At $t = 0$, system = rocket + fuel.

After a time τ, system is shown below



Initial linear momentum $= m_0 v_0$.

Final linear momentum of rocket & unburned fuel is $m(\tau)v(\tau)$.

Fuel ejected in time $[t, t+dt]$

$$(-\dot{m}(t))dt$$

with forward velocity at instant of ejection is $v(t) - u$

$\therefore$ linear mom. of eject. fuel

$$= -\dot{m}(v-u)\,dt.$$

$\therefore$ Total linear mom. of fuel ejected in time interval $[0, \tau]$ is,

$$-\int_0^\tau \dot{m}(v-u)\,dt.$$

$\therefore$ Conser. of momentum

$$\Rightarrow m_0 v_0 = m(\tau)v(\tau) - \int_0^\tau \dot{m}(v-u)\,dt.$$

$$\Rightarrow \int_0^\tau \left[\frac{d}{dt}(mv) - \dot{m}(v-u)\right]dt = 0.$$

Must hold for any τ during burn,

$$\Rightarrow \frac{d}{dt}(mv) - \dot{m}(v-u) = 0, \quad 0 \leq t \leq T.$$

$$\Rightarrow \quad m\frac{dv}{dt} = (-\dot{m})u.$$

$$\Rightarrow \int dv = \int \frac{(-\dot{m})u}{m}\, dt$$

$$= -u\int \frac{dm}{m} = -u\ln m + C_1.$$

At $t=0$, $v=v_0$, $m=m_0$.

$$\Rightarrow \quad v(t) = v_0 + u\ln\left(\frac{m_0}{m(t)}\right).$$

Let fuel burns at a constant rate

$\frac{dM}{dt} = b$ & it lasts for time T.

If mass of vehicle is M_v & mass of fuel is M_f at $t=0$.

$$\Rightarrow \quad M_0 = M_v + M_f.$$

$$M(t) = M_v + M_f\left(1-\frac{t}{T}\right).$$

$$= M_0 - M_f\frac{t}{T}, \quad 0 \le t \le T.$$

& $M(t) = M_v$ for $t \ge T$.

$$\Rightarrow v = \frac{dx}{dt} = v_0 - u \ln\left(1 - \frac{M_f}{m_0}\frac{t}{T}\right).$$

$$\therefore\ x = x_0 + v_0 t - u\int_0^t \ln\left(1 - \frac{M_f}{m_0}\frac{t}{T}\right) dt$$

$$\Rightarrow x = x_0 + v_0 t - u\left[\left(t - \frac{m_0 T}{M_f}\right)\ln\left(1 - \frac{M_f}{m_0}\frac{t}{T}\right) - t\right]$$

$$v_{max} = v(t)$$

$$= v_0 + u \ln\left(\frac{M_0}{M_v}\right).$$

$$= \cancel{v_0 + u \ln}$$

$$= v_0 + u \ln\left(1 + \frac{M_f}{m_v}\right).$$

<u>Under gravity</u>

$$M\dot{v} = -u\frac{dM}{dt} - Mg.$$

$$\Rightarrow \dot{v} = -\frac{u}{M}\frac{dm}{dt} - g.$$

$$\Rightarrow v = -u \ln\left(\frac{m(t)}{m_0}\right) - gt.$$

Assume $x = 0$ & $v = v_0 = 0$ at $t = 0$.

$$x = ut - \frac{1}{2}gt^2 - \left(t - \frac{m_0 T}{M_f}\right)\ln\left(1 - \frac{M_f}{M_0}\frac{t}{T}\right).$$