Beats

The variation in pressure (relative to avg. background pressure) due to a source (like a tuning fork struck at frequency ω_1) results in displacement given by

$$x_1 = \text{Re } z_1 \quad , \text{ where } z_1 = Ae^{i\omega_1 t}.$$

For another tuning fork struck at ω_2 (same amplitude),

$$x_2 = \text{Re } z_2 \quad , \text{ where } z_2 = Ae^{i\omega_2 t}.$$

$$\therefore \quad z = z_1 + z_2 = A(e^{i\omega_1 t} + e^{-i\omega_2 t}).$$

Define, $\omega_e = \dfrac{\omega_1 - \omega_2}{2}$

& $\omega_{av} = \dfrac{\omega_1 + \omega_2}{2}$

$$\Rightarrow \quad \omega_1 = \omega_{av} + \omega_e$$

$$\& \quad \omega_2 = \omega_{av} - \omega_e.$$

$$\therefore \quad z = A\left[e^{i(\omega_{av} + \omega_e)t} + e^{i(\omega_{av} - \omega_e)t}\right].$$

$$= 2Ae^{i\omega_{av}t} \cos \omega_e t.$$

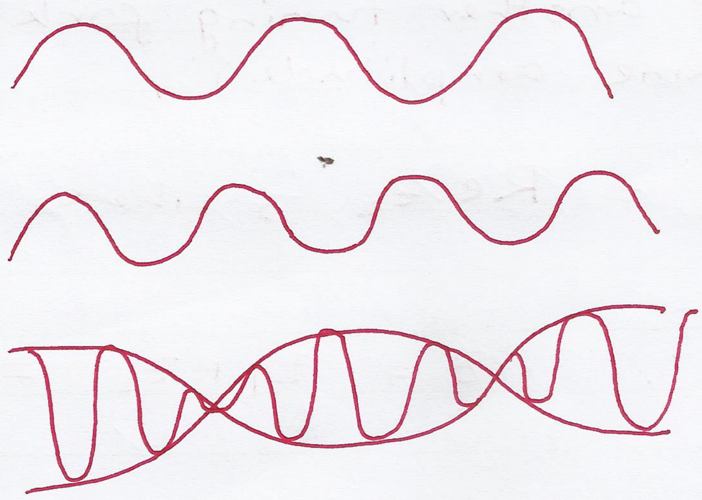
$$\therefore \quad x = \text{Re } z = 2A \cos \omega_e t \cos \omega_{av} t$$

Slow oscillations due to the transition from constructive to destructive interference & back: an envelope function

$$x = 2A \cos \omega_e t \cos \omega_{av} t$$

$2A$: at constructive interference maxima

$\cos \omega_{av} t$: rapid oscillations

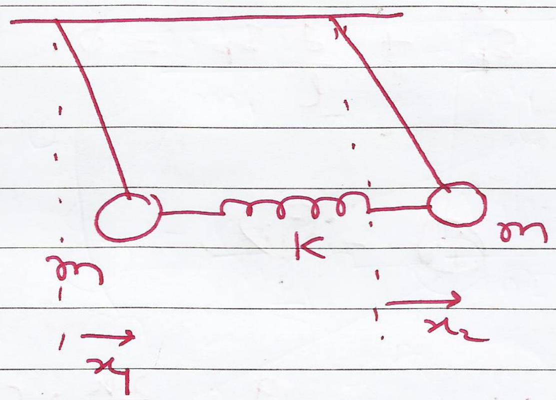


Beat period = period between maximum amplitudes

$$\text{i.e.,} \quad T_{beat} = \frac{\pi}{\omega_e} = \frac{2\pi}{\omega_1 - \omega_2}$$

$$\therefore \quad f_{beat} = \frac{1}{T_{beat}} = 2\left(\frac{f_1 - f_2}{2}\right) = f_1 - f_2$$

Coupled pendula



$$m\ddot{x}_1 = -\frac{mg}{l}x_1 - k(x_1 - x_2).$$

$$m\ddot{x}_2 = -\frac{mg}{l}x_2 - k(x_2 - x_1).$$

i.e., $$\ddot{x}_1 + \frac{g}{l}x_1 + \frac{k}{m}(x_1 - x_2) = 0. \quad \text{—(i)}$$

$$\ddot{x}_2 + \frac{g}{l}x_2 + \frac{k}{m}(x_2 - x_1) = 0. \quad \text{—(ii)}$$

Use
$$\left.\begin{aligned} S_p &= \frac{x_1 + x_2}{\sqrt{2}} \\ S_b &= \frac{x_1 - x_2}{\sqrt{2}} \end{aligned}\right\} \quad \text{—(iii)}$$

Adding (i) & (ii)

$$\Rightarrow \quad \ddot{x}_1 + \ddot{x}_2 + \frac{g}{l}(x_1 + x_2) = 0.$$

$$\therefore \quad \ddot{S}_p + \frac{g}{l} S_p = 0.$$

Similarly (i) − (ii),

$$\Rightarrow \quad \ddot{S}_b + \left(\frac{g}{l} + \frac{2k}{m}\right) S_b = 0.$$

$\therefore$ we can define

$$\omega_p = \sqrt{\frac{g}{l}}$$

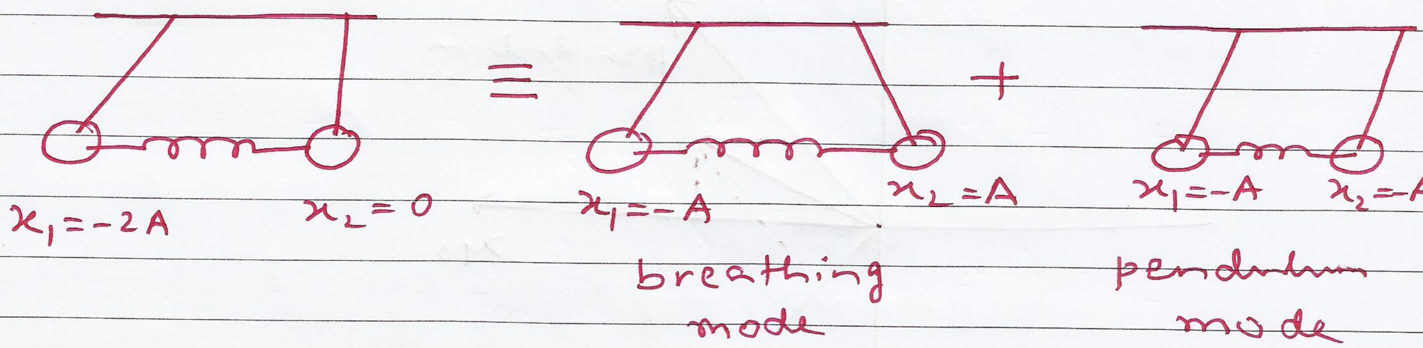
$$\& \quad \omega_b = \sqrt{\frac{g}{l} + \frac{2k}{m}}$$

& called the pendulum mode (in-phase)

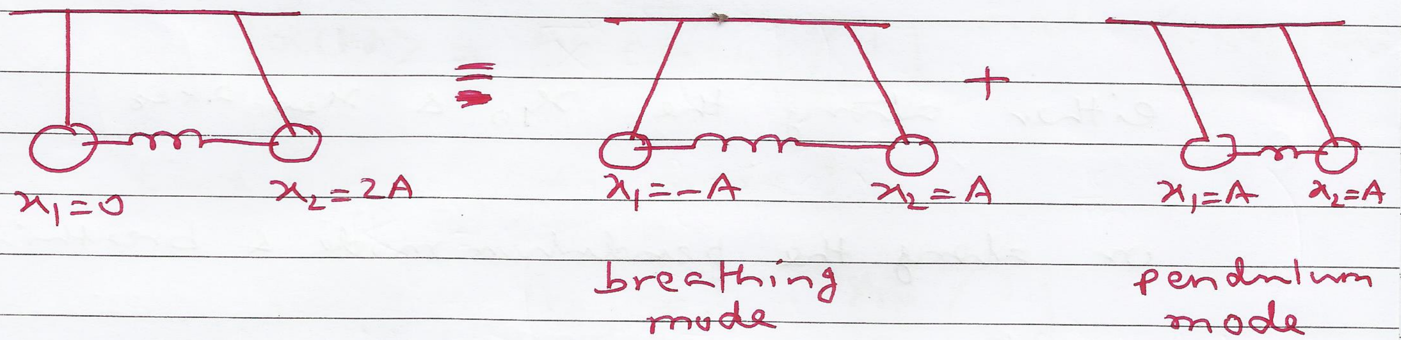
& the breathing mode (out of phase)

respectively.

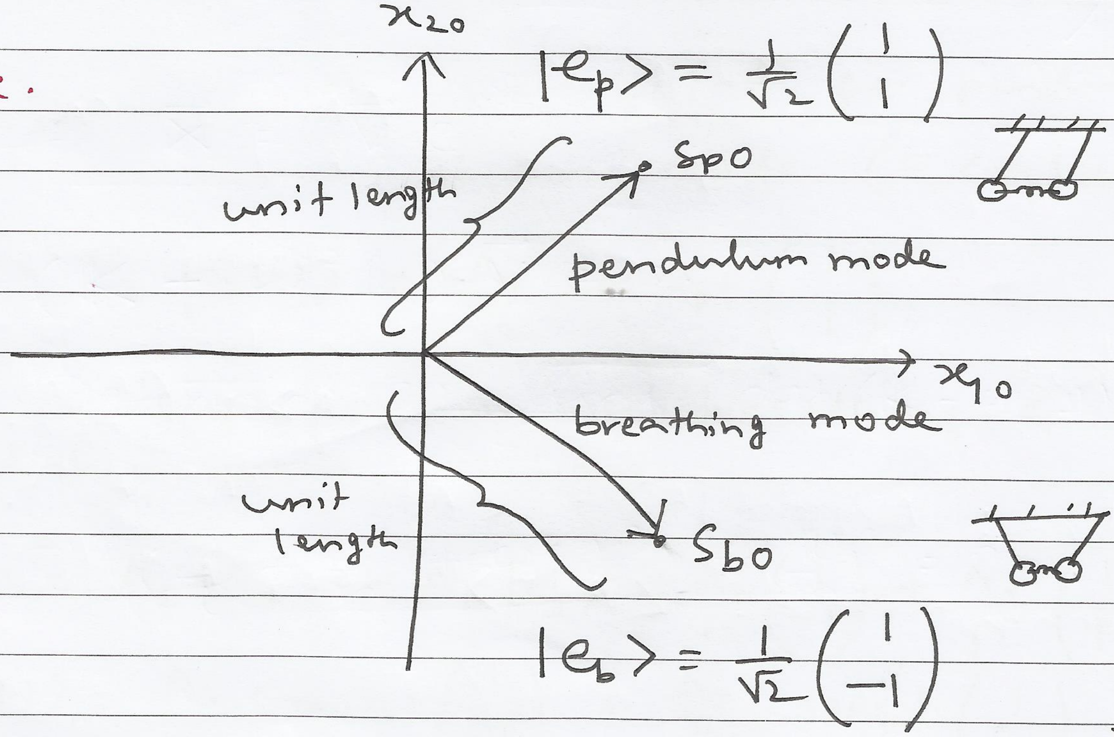
Note : A beating phenomena exist due to superposition of pendulum mode & breathing mode

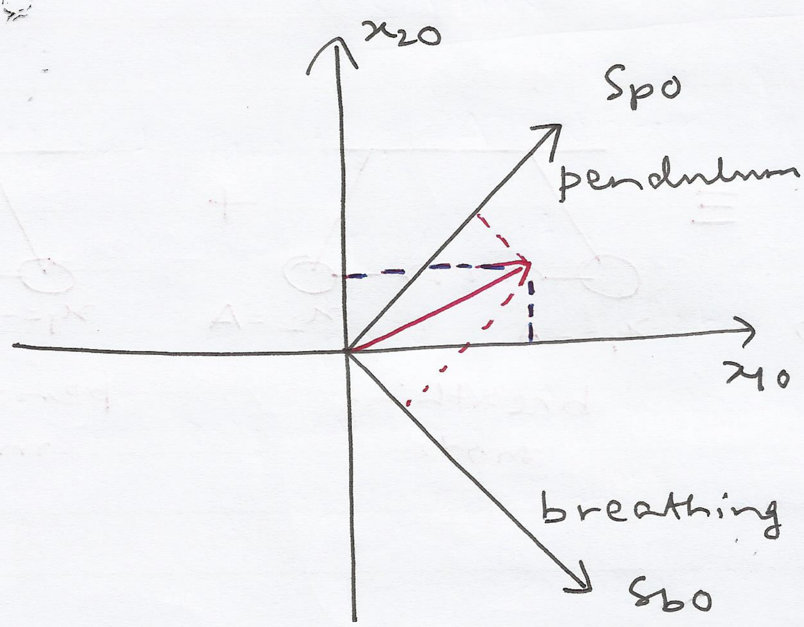


Similarly,



etc.





The red vector can be resolved either along the x_{10} & x_{20} axes or along the pendulum mode & breathing mode axes.

Notation

$$|x(t)\rangle = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

(I.) For pure breathing mode:

$$|x_0\rangle = A_b|e_b\rangle = \frac{A_b}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

A_b is amplitude for breathing mode. $(\equiv \langle e_b|x_0\rangle)$

$$\therefore |x(t)\rangle = \frac{A_b}{\sqrt{2}} \cos\omega_b t \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

(II.) For pure pendulum mode:

$$|x_0\rangle = A_p|e_p\rangle = \frac{A_p}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

A_p is amplitude for pendulum mode. $(\equiv \langle e_p|x_0\rangle)$.

$$\therefore |x(t)\rangle = \frac{A_p}{\sqrt{2}} \cos\omega_p t \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

(III.) For a superposition of breathing & pendulum mode, we have,

$$|x_0\rangle = A_b|e_b\rangle + A_p|e_p\rangle = \frac{A_b}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{A_p}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

$$\therefore |x(t)\rangle = \frac{A_b}{\sqrt{2}} \cos\omega_b t \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{A_p}{\sqrt{2}} \cos\omega_p t \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Some definitions

Hilbert space: A vector space with a defined inner product.

"a space in which each point represents a particular configuration of the system".

Note: In most applications, ~~each~~ space has infinite dimensions ∴ each point represents a function.

Column vector: Matrix of form $\begin{pmatrix} A_1 \\ A_2 \\ \vdots \end{pmatrix}$ that represents a vector.

Adjoint (Hermitian transpose or Hermitian conjugate):

Adjoint of $\begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$ is $(A_1^* \; A_2^*)$.

$*$ indicates complex conjugation.

Ket: Another name for vector in Hilbert space
$|A\rangle = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$.

Bra: Adjoint $|A\rangle = \langle A| \equiv (A_1^* \; A_2^*)$.

Inner product: $\langle A|B\rangle = (A_1^* \; A_2^*)\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = A_1^* B_1 + A_2^* B_2$

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Normalized vector: A vector in Hilbert space with length 1 (norm 1 or inner product with its adjoint 1).
Note: Inner product of normalized vector with itself is 1.

Complete basis: A set of vectors, linear combination of which can be used to create any vector in the Hilbert space.
For a 2-d Hilbert space, any two vectors that are ~~non parrat~~ non-parallel would form a ~~thi~~ complete basis.

Complete orthonormal basis: A set of orthogonal normalized vectors that form a complete basis.