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Normal modes (contd).

Recipe (system of coupled oscillators with N masses)

- Write $F^{j}_{tot} = m_j \ddot{x}_j$ for each object.

This leads to a system on N coupled differential equations.

- Ansatz for normal mode solution,

$$x_j \equiv \mathrm{Re}[X_j e^{i\omega t}].$$

ω is frequency of the normal mode.
X_j is amplitude of oscillation of j^{th} mass.

Convention : At $t = 0$
overall phase factor $\varphi = 0$.
$\therefore$ Amplitude X_j of object j is equal to the initial position x_{j0} of object j.

- Plug the ansatz ; $e^{i\omega t}$ is common on both l.h.s & r.h.s & cancels out, leading to N coupled linear equations involving the variables X_j.

- Express the equations as an eigenvalue equation, i.e.,

$$\overleftrightarrow{A}\,|e_u\rangle = \omega^2\,|e_u\rangle$$

where, $\overleftrightarrow{A}$ is an $N \times N$ matrix

& $|e_u\rangle$ is the column vector $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix}$.

Note: In our case (as found in class), $\overleftrightarrow{A} = \overleftrightarrow{M}^{-1}\,\overleftrightarrow{\mathbb{K}}$.

- Rearrange the equations to obtain "characteristic equation",

$$\overleftrightarrow{B}\,|e_u\rangle = 0$$

Note: $\overleftrightarrow{B} = (M^{-1}\mathbb{K} - \omega^2 \mathbb{1})$

where $\mathbb{1}$ is $N \times N$ identity matrix.

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- Eigenvalues ω^2 are obtained using,

$$\det \overleftrightarrow{B} = 0$$

i.e., $\det\left(\overleftrightarrow{M}^{-1}\overleftrightarrow{K} - \omega^2 \mathbb{1}\right) = 0.$

leading to Nth-order equation for ω^2, with N solutions.

- To obtain eigenvectors, for each of the values of ω^2, use

$\overleftrightarrow{B}\,|e_n\rangle = 0$ & solve for $x_2, x_3, \ldots$ in terms of x_1.

- $|e_n\rangle$ is normalized s.t., $\langle e_n|e_n\rangle = 1$.

Physical interpretation

ω_n is frequency of the n^{th} normal mode

$|e_n\rangle = \begin{pmatrix} x_1 \\ \vdots \end{pmatrix}$ is a vector of length 1

specifying the initial positions of the masses (with zero initial velocities) for n^{th} normal mode

Any state of the system, $|x(t)\rangle = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \end{pmatrix}$ can be written as

$$|x(t)\rangle = \mathrm{Re}\left[\sum_{n=1}^{N} C_n e^{i\omega_n t} |e_n\rangle\right].$$

Oddy

Treatment for equal masses

Initial positions of the objects are given by

$$|x_0\rangle = \begin{pmatrix} x_{10} \\ x_{20} \\ \vdots \end{pmatrix} = \mathrm{Re}\left[\sum_{n=1}^{N} C_n |e_n\rangle\right]$$

(at $t=0$).

& initial velocities are given by

$$|\dot{x}_0\rangle = \begin{pmatrix} \dot{x}_{10} \\ \dot{x}_{20} \\ \vdots \end{pmatrix} = \frac{d}{dt}\left[|x(t)\rangle\right]_{t=0}$$

$$= \frac{d}{dt}\left[\mathrm{Re}\left[\sum_{n=1}^{N} C_n e^{i\omega_n t} |e_n\rangle\right]\right]_{t=0}$$

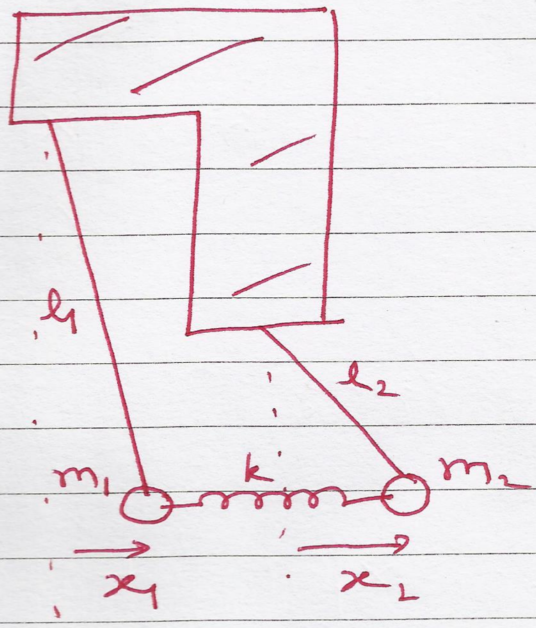
$$= \mathrm{Re}\left[\sum_{n=1}^{N} i\omega_n C_n |e_n\rangle\right].$$

But $C_n = \mathrm{Re}(C_n) + i\,\mathrm{Im}(C_n)$.

$$\Rightarrow |x_0\rangle = \sum_{n=1}^{N} \mathrm{Re}(C_n)|e_n\rangle.$$

$$\& \quad |\dot{x}_0\rangle = -\sum_{n=1}^{N} \omega_n \,\mathrm{Im}(C_n)|e_n\rangle.$$

Unequal length (coupled pendula)



$$m_1 \ddot{x}_1 = -\frac{m_1 g}{l_1} x_1 - k(x_1 - x_2) \quad \text{——(i)}$$

$$m_2 \ddot{x}_2 = -\frac{m_2 g}{l_2} x_2 - k(x_2 - x_1). \quad \text{——(ii)}$$

Rearranging,

$$\ddot{x}_1 + \overbrace{\left(\frac{g}{l_1} + \frac{k}{m_1}\right)}^{\omega_A^2} x_1 - \frac{k}{m_1} x_2 = 0. \quad \text{——(iii)}$$

$$\ddot{x}_2 + \underbrace{\left(\frac{g}{l_2} + \frac{k}{m_2}\right)}_{\omega_B^2} x_2 - \frac{k}{m_2} x_1 = 0. \quad \text{——(iv)}$$

Most general way to represent it is,

$$\left.\begin{array}{l} x_1 = \text{Re}\, z_1 \\ x_2 = \text{Re}\, z_2 \end{array}\right\} \text{for the motion}$$

where, $z_1 \equiv X_1 e^{i\omega t} e^{i\varphi}$

$z_2 \equiv X_2 e^{i\omega t} e^{i\varphi}$

Suitable choice of $t = 0$, we can arrange to have $\varphi = 0$. Thus, $z_1 \equiv X_1 e^{i\omega t}$, $z_2 \equiv X_2 e^{i\omega t}$

When $t = 0$, $X_1 = x_{10}$, $X_2 = x_{20}$
(physical interpretation).

Plugging into (iii) & (iv),

$$\Rightarrow \; -\omega^2 X_1 e^{i\omega t} + \omega_A^2 X_1 e^{i\omega t} - \frac{k}{m_1} X_2 e^{i\omega t} = 0$$

$$\Rightarrow \; \omega_A^2 X_1 - \frac{k}{m_1} X_2 = \omega^2 X_1 .$$

Similarly,

$$\omega_B^2 X_2 - \frac{k}{m_2} X_2 = \omega^2 X_2 .$$

$$\text{i.e.,} \quad \underbrace{\begin{pmatrix} \omega_A^2 & -\frac{k}{m_1} \\ -\frac{k}{m_2} & \omega_B^2 \end{pmatrix}}_{\text{matrix } \overleftrightarrow{A}} \underbrace{\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}}_{\text{Eigenvector}} = \underbrace{\omega^2}_{\text{Eigenvalue}} \underbrace{\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}}_{\text{Eigenvector}}$$

$$\Downarrow$$

$$\overleftrightarrow{B} \Rightarrow \begin{pmatrix} \omega_A^2 - \omega^2 & -\frac{k}{m_1} \\ -\frac{k}{m_2} & \omega_B^2 - \omega^2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = 0 . \quad \Leftarrow |e_\omega\rangle$$

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$\det \overleftrightarrow{B} = 0$

leads to an equation for ω^2 which can be solved ("Tutorial") to yield,

$$\omega^2 = \frac{\omega_A^2 + \omega_B^2 \pm \sqrt{(\omega_A^2 - \omega_B^2)^2 + \frac{4k^2}{m_1 m_2}}}{2}.$$

$$\begin{cases} \omega_A^2 = \frac{g}{L_1} + \frac{k}{m_1} \\ \omega_B^2 = \frac{g}{L_2} + \frac{k}{m_2} \end{cases}$$

& solving for eigenvectors lead to

$|e_{ub}\rangle \equiv \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}$ "Breathing".

& $|e_{up}\rangle \equiv \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}$ "Pendulum"

(not normalized).

corresponding normalized eigenvectors are,

$|e_{ub}\rangle = \begin{pmatrix} 1/2 \\ -\sqrt{3}/2 \end{pmatrix}$

$|e_{up}\rangle = \begin{pmatrix} 1/2 \\ \sqrt{3}/2 \end{pmatrix}$

Oddy